Orthogonal and symplectic matrix models: universality and other properties.

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Abstract

We study orthogonal and symplectic matrix models with polynomial potentials and multi interval supports of the equilibrium measure. For these models we find the bounds (similar to the case of hermitian matrix models) for the rate of convergence of linear eigenvalue statistics and for the variance of linear eigenvalue statistics and find the logarithms of partition functions up to the order $O(1)$. We prove also universality of local eigenvalue statistics in the bulk.

1 Introduction and main results

In this paper we consider ensembles of random matrices, whose joint eigenvalues distribution is

$$p_{n,\beta}(\lambda_1,...,\lambda_n)=Q_{n,\beta}^{-1}[V]\prod_{i=1}^n e^{-n\beta V(\lambda_i)/2}\prod_{1\le i<j\le n}|\lambda_i-\lambda_j|^\beta=Q_{n,\beta}^{-1}e^{\beta H(\lambda_1,...,\lambda_n)/2}, \qquad (1.1)$$

where the function H, which we call Hamiltonian to stress the analogy with statistical mechanics, and the normalizing constant $Q_{n,\beta}[V]$ have the form

$$\begin{aligned} H(\lambda_1,\dots,\lambda_n) &= -n\sum_{i=1}^n V(\lambda_i)+\sum_{i\ne j}\log|\lambda_i-\lambda_j|,\\ Q_{n,\beta}[V] &= \int e^{\beta H(\lambda_1,...,\lambda_n)/2}d\lambda_1\dots d\lambda_n. \end{aligned} \qquad (1.2)$$

The function V, called the potential, is a real valued Hölder function satisfying the condition

$$V(\lambda)\ge 2(1+\epsilon)\log(1+|\lambda|). \qquad (1.3)$$

This distribution can be considered for any $\beta>0$, but the cases $\beta=1,2,4$ are especially important, since they correspond to real symmetric, hermitian, and symplectic matrix models respectively.

We will consider also the marginal densities of (1.1) (correlation functions)

$$p_{l,\beta}^{(n)}(\lambda_1,...,\lambda_l)=\int_{\mathbb{R}^{n-l}}p_{n,\beta}(\lambda_1,...\lambda_l,\lambda_{l+1},...,\lambda_n)d\lambda_{l+1}...d\lambda_n, \qquad (1.4)$$

and denote

$$\mathbf{E}_\beta\{(\dots)\}=\int(\dots)p_{n,\beta}(\lambda_1,...,\lambda_n)d\lambda_1,\dots d\lambda_n. \qquad (1.5)$$

It is known (see [2, 12]) that if V' is a Hölder function, then the first marginal density $p_{1,\beta}^{(n)}$ converges weakly to the function ρ (equilibrium density) with a compact support σ. The density ρ maximizes the functional, defined on the class $\mathcal{M}_1$ of positive unit measures on $\mathbb{R}$

$$\mathcal{E}_V(\rho) = \max_{m\in\mathcal{M}_1}\left\{L[\,dm,dm] - \int V(\lambda)m(d\lambda)\right\} = \mathcal{E}[V], \tag{1.6}$$

where we denote

$$L[\,dm,dm] = \int \log|\lambda-\mu|dm(\lambda)dm(\mu), \quad L[f,g] = \int \log|\lambda-\mu|f(\lambda)g(\mu)d\lambda d\mu. \tag{1.7}$$

The support σ and the density ρ are uniquely defined by the conditions:

$$\begin{array}{ll} v(\lambda) := 2\displaystyle\int \log|\mu-\lambda|\rho(\mu)d\mu - V(\lambda) = \sup v(\lambda) := v^*, & \lambda\in\sigma \\ v(\lambda) \le \sup v(\lambda), \quad \lambda\notin\sigma, & \sigma = \mathrm{supp}\{\rho\}. \end{array} \tag{1.8}$$

For $\beta = 2$ it is well known (see [14]) that all correlation functions (1.4) can be represented as

$$p_{l,\beta}^{(n)}(\lambda_1,...,\lambda_l) = \frac{(n-l)!}{n!}\det\{K_{n,2}(\lambda_j,\lambda_k)\}_{j,k=1}^{l}, \tag{1.9}$$

where

$$K_{n,2}(\lambda,\mu) = \sum_{l=0}^{n-1}\psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu). \tag{1.10}$$

This function is known as a reproducing kernel of the orthonormalized system

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\}p_l^{(n)}(\lambda), \;\; l=0,..., \tag{1.11}$$

in which $\{p_l^{(n)}\}_{l=0}^{n}$ are orthogonal polynomials on $\mathbb{R}$ associated with the weight $w_n(\lambda) = e^{-nV(\lambda)}$, i.e.,

$$\int p_l^{(n)}(\lambda)p_m^{(n)}(\lambda)w_n(\lambda)d\lambda = \delta_{l,m}. \tag{1.12}$$

The orthogonal polynomial machinery, in particular, the Christoffel-Darboux formula and Christoffel function simplify considerably the studies of marginal densities (1.4). This allows to study the local eigenvalue statistics in many different cases: bulk of the spectrum, edges of the spectrum, special points, etc. (see [16], [17],[5],[3],[4],[19],[11]).

For $\beta = 1,4$ the situation is more complicated. It was shown in [22] that all correlation functions can be expressed in terms of some matrix kernels (see (1.13) – (1.17) below). But the representation is less convenient than (1.9) – (1.10). It makes difficult the problems, which for $\beta = 2$ are just simple exercises. For example, the bound for the variance of linear eigenvalue statistics (1.21) for $\beta = 1,4$ till now was known only for one interval σ (see [12]), while for $\beta = 2$ it is a trivial corollary of the Christoffel-Darboux formula for any σ.

The matrix kernels for $\beta = 1,4$ have the form

$$K_{n,1}(\lambda,\mu) := \begin{pmatrix} S_{n,1}(\lambda,\mu) & -\frac{\partial}{\partial\mu}S_{n,1}(\lambda,\mu) \\ (\epsilon S_{n,1})(\lambda,\mu) - \epsilon(\lambda-\mu) & S_{n,1}(\mu,\lambda)\end{pmatrix} \text{ for } \beta=1, n \text{ even}, \tag{1.13}$$

$$K_{n,4}(\lambda,\mu) := \begin{pmatrix} S_{n,4}(\lambda,\mu) & -\frac{\partial}{\partial\mu}S_{n,4}(\lambda,\mu) \\ (\epsilon S_{n,4})(\lambda,\mu) & S_{n,4}(\mu,\lambda)\end{pmatrix} \text{ for } \beta=4, \tag{1.14}$$

where

$$S_{n,1}(\lambda,\mu)=-\sum_{j,k=0}^{n-1}\psi_j^{(n)}(\lambda)(M_n^{(n)})^{-1}_{jk}(\epsilon\psi_k^{(n)})(\mu), \tag{1.15}$$

$$S_{n/2,4}(\lambda,\mu)=-\sum_{j,k=0}^{n-1}(\psi_j^{(n)})'(\lambda)(D_n^{(n)})^{-1}_{jk}\psi_k^{(n)}(\mu), \tag{1.16}$$

$\epsilon(\lambda)=\frac{1}{2}\mathrm{sgn}(\lambda)$, sgn denotes the standard signum function,

$$(\epsilon f)(\lambda):=\int_{\mathbb{R}}\epsilon(\lambda-\mu)f(\mu)\,d\mu d\lambda'.$$

$D_n^{(n)}$ and $M_n^{(n)}$ in (1.15) and (1.16) are the left top corner $n\times n$ blocks of the semi-infinite matrices that correspond to the differentiation operator and to some integration operator respectively.

$$D_\infty^{(n)}:=\left(\left(\psi_j^{(n)}\right)',\psi_k^{(n)}\right)_{j,k\ge0},\quad D_n^{(n)}=\{D_{jk}^{(n)}\}_{j,k=0}^n, \tag{1.17}$$
$$M_\infty^{(n)}:=\left(\epsilon\psi_j^{(n)},\psi_k^{(n)}\right)_{j,k\ge0},\quad M_n^{(n)}=\{M_{jk}^{(n)}\}_{j,k=0}^n.$$

Both matrices $D_\infty^{(n)}$ and $M_\infty^{(n)}$ are skew-symmetric, and since $\epsilon(\psi_j^{(n)})'=\psi_j^{(n)}$, we have for any $j,l\ge0$ that

$$\delta_{jl}=(\epsilon(\psi_j^{(n)})',\psi_l)=\sum_{k=0}^{\infty}(D_\infty^{(n)})_{jk}(M_\infty^{(n)})_{kl}\quad\Longleftrightarrow\quad D_\infty^{(n)}M_\infty^{(n)}=1=M_\infty^{(n)}D_\infty^{(n)}.$$

It was observed in [23] that, if V is a rational function, in particular, a polynomial of degree $2m$, then the kernels $S_{n,1},S_{n,4}$ can be written as

$$S_{n,1}(\lambda,\mu)=K_{n,2}(\lambda,\mu)+n\sum_{j,k=-(2m-1)}^{2m-1}F_{jk}^{(1)}\psi_{n+j}^{(n)}(\lambda)\epsilon\psi_{n+k}^{(n)}(\mu), \tag{1.18}$$
$$S_{n/2,4}(\lambda,\mu)=K_{n,2}(\lambda,\mu)+n\sum_{j,k=-(2m-1)}^{2m-1}F_{jk}^{(4)}\psi_{n+j}^{(n)}(\lambda)\epsilon\psi_{n+k}^{(n)}(\mu),$$

where $F_{jk}^{(1)}$, $F_{jk}^{(4)}$ can be expressed in terms of the matrix T_n^{-1}, where T_n is the $(2m-1)\times(2m-1)$ block in the bottom right corner of $D_n^{(n)}M_n^{(n)}$, i.e.,

$$(T_n)_{jk}:=(D_n^{(n)}M_n^{(n)})_{n-2m+j,n-2m+k},\qquad 1\le j,k\le 2m-1. \tag{1.19}$$

The main technical obstacle to study the kernels $S_{n,1},S_{n,4}$ is the problem to prove that $(T_n^{-1})_{jk}$ are bounded uniformly in n. Till now this technical problem was solved only in a few cases. In the papers [6, 7] the case $V(\lambda)=\lambda^{2m}(1+o(1))$ (in our notations) was studied and the problem of invertibility of T_n was solved by computing the entries of T_n explicitly. Similar method was used in [8] to prove bulk and edge universality (including the case of hard edge) for the Laguerre type ensembles with monomial V. In the paper [21] the problem of invertibility of T_n was solved also by computing the entries of T_n for V being an even quatric polynomial. In [19, 20] similar problem was solved without explicit computation of

the entries of T_n, it was shown that for any real analytic V with one interval support of the equilibrium density $(M_n^{(n)})^{-1}$ is uniformly bounded in the operator norm. This allowed us to prove bulk and edge universality for $\beta = 1$ in the one interval case.

But there is also a possibility to prove that T_n is invertible with another technique. As a by product of the calculation in [22] one also obtains relations between the partition functions $Q_{n,\beta}$ and the determinants of $M_n^{(n)}$ and $D_n^{(n)}$:

$$\det M_n^{(n)} = \left(\frac{Q_{n,1}\Gamma_n}{n!2^{n/2}}\right)^2 \quad , \quad \det D_n^{(n)} = \left(\frac{Q_{n/2,4}\Gamma_n}{(n/2)!2^{n/2}}\right)^2,$$

where $\Gamma_n := \prod_{j=0}^{n-1}\gamma_j^{(n)}$, and $\gamma_j^{(n)}$ is the leading coefficient of $p_j^{(n)}(\lambda)$ of (1.12). It is also known (see [14]) that $Q_{n,2} = \Gamma_n^2/n!$. Since $D_\infty^{(n)}M_\infty^{(n)} = 1$ and $(D_\infty^{(n)})_{jk} = 0$ for $|j-k| > 2m-1$ (see (3.3)), we have $D_n^{(n)}M_n^{(n)} = 1 + \Delta_n$ with Δ_n being zero except for the bottom $2m-1$ rows, and we arrive at a formula, first observed in [21]:

$$\det(T_n) = \det(D_n^{(n)}M_n^{(n)}) = \left(\frac{Q_{n,1}Q_{n/2,4}}{Q_{n,2}(n/2)!2^n}\right)^2. \tag{1.20}$$

Hence to control $\det(T_n)$, it suffices to control $\log Q_{n,\beta}$ for $\beta = 1, 2, 4$ up to the order $O(1)$. In the paper [15] the corresponding expansion of $\log Q_{n,\beta}$ was constructed by using some generalization of the method of [12]. The original method was proposed to study the fluctuations of linear eigenvalue statistics

$$\mathcal{N}_n[\varphi] = \sum_{i=1}^{n}\varphi(\lambda_i), \tag{1.21}$$

in particular, to control the expectation and the variance of $n^{-1}\mathcal{N}_n[\varphi]$ up to the terms $O(n^{-2})$ for any β, but only in the case of one interval support of the equilibrium measure and polynomial V, satisfying some additional assumption. The method was used also to prove CLT for fluctuations of $\mathcal{N}_n[\varphi]$. In the paper [15] the method of [12] was simplified, that allowed to generalize it on the case of real analytic V with one interval support of ρ, without any other assumption. Unfortunately, there is no hope to generalize the method of [12, 15] on the case of multi interval support σ directly, because the method is based on solving of some integral equation (see Eq.(2.12) below) which is not uniquely solvable in the case of multi interval support.

In the present paper the problem to control $Q_{n,\beta}[V]$ for $\beta = 1, 2, 4$ is solved in a little bit different way. We prove that for analytical potential V with q-interval support σ of the equilibrium density $Q_{n,\beta}$ can be factorized to a product of $Q_{k_\alpha^*,\beta}[V_a^{(\alpha)}]$, $\alpha = 1, \dots, q$, where $k_\alpha^* \sim \mu_\alpha^* n$ (see (1.33)), and the "effective potentials" $V_\alpha^{(a)}$ (see (1.35)) are defined in terms of σ, V and ρ.

To be more precise let us formulate our main conditions.

Condition C1. *V is a polynomial of degree $2m$ with positive leading coefficient, and the support of its equilibrium measure is*

$$\sigma = \bigcup_{\alpha=1}^{q}\sigma_\alpha, \quad \sigma_\alpha = [E_{2\alpha-1}, E_{2\alpha}] \tag{1.22}$$

Condition C2. *The equilibrium density ρ can be represented in the form*

$$\rho(\lambda) = \frac{1}{2\pi}P(\lambda)\Im X^{1/2}(\lambda + i0), \quad \inf_{\lambda\in\sigma}|P(\lambda)| > 0, \tag{1.23}$$

where

$$X(z)=\prod_{\alpha=1}^{2q}(z-E_\alpha), \tag{1.24}$$

and we choose a branch of $X^{1/2}(z)$ such that $X^{1/2}(z)\sim z^q$, as $z\to+\infty$. Moreover, the function v defined by (1.8) attains its maximum only if λ belongs to σ.

Remark 1 *It is known (see, e.g., [1]) that for analytic V the equilibrium density ρ has the form (1.23) – (1.24) with $P\ge 0$. The function P in (1.23) is analytic and can be represented in the form*

$$P(z)=\frac{1}{2\pi i}\oint_{\mathcal{L}}\frac{V'(z)-V'(\zeta)}{(z-\zeta)X^{1/2}(\zeta)}d\zeta. \tag{1.25}$$

Hence condition C2 means that ρ has no zeros in the internal points of σ and behaves like square root near the edge points. This behavior of V is usually called generic.

We will use also the notations

$$\begin{aligned}&\sigma_\varepsilon=\bigcup_{\alpha=1}^{q}\sigma_{\alpha,\varepsilon},\quad \sigma_{\alpha,\varepsilon}=[E_{2\alpha-1}-\varepsilon,E_{2\alpha}+\varepsilon],\\ &\mathrm{dist}\,\{\sigma_{\alpha,\varepsilon},\sigma_{\alpha',\varepsilon}\}>\delta>0,\quad \alpha\ne\alpha'.\end{aligned} \tag{1.26}$$

The first result of the paper is the theorem which allows us to control $\log Q_{n,\beta}$ in the one interval case up to $O(1)$ terms. Since the paper [2] it is known that

$$\log Q_{n,\beta}[V]=\beta n^2\mathcal{E}[V]/2+O(n\log n).$$

But, as it was discussed above, for many problems it is important to control the next terms of asymptotic expansion of $\log Q_{n,\beta}$ (see also the discussion in [9], where the expansion in n^{-1} was constructed for $\beta=2$ and V being a polynomial, close in a certain sense to $V_0(\lambda)=\lambda^2/2$.)

We would like to note that almost all assertions of Theorem 1 below were obtained in [15]. The difference is that here we need to control that the remainder bounds are uniform in some parameter η, which we put in front of V in H (see (1.2)). Note also that it is important for us that here V may be non polynomial function analytic only in a some open domain $\mathbf{D}\subset\mathbb{C}$ containing σ.

We will use below the Stieltjes transform, defined for an integrable function p as

$$g(z)=\int\frac{p(\lambda)d\lambda}{\lambda-z}. \tag{1.27}$$

Theorem 1 *Let V satisfy (1.3), the equilibrium density ρ (see (1.8)) have the form (1.23) with $q=1$, and $\sigma=\mathrm{supp}\,\rho=[a,b]$. Assume also that V is analytic in the domain $\mathbf{D}\supset\sigma_\varepsilon$. Consider the distribution (1.1) with V replaced by ηV. Then there exists $\varepsilon_1>0$ such that for any $\eta:|\eta-1|\le\varepsilon_1$ we have:*
(i) The Stieltjes transform $g_n^{(\eta)}(z)$ (1.27) of the first marginal $p_{1,\beta}^{(n,\eta)}$ of (1.1) for z such that $d(z):=\mathrm{dist}\,\{z,\sigma_\varepsilon\}\ge n^{-1/6}\log n$ has the form

$$\begin{aligned}g_n^{(\eta)}(z)&=g_\eta(z)+n^{-1}u_{n,\eta}(z),\\ u_{n,\eta}(z)&=\left(\frac{2}{\beta}-1\right)\frac{1}{2\pi i X_\eta^{1/2}(z)}\oint_{\mathcal{L}}\frac{g_\eta'(\zeta)d\zeta}{P_\eta(\zeta)(z-\zeta)}+n^{-1}O(d^{-10}(z)),\end{aligned} \tag{1.28}$$

where $g_\eta(z)$ is the Stieltjes transform of the equilibrium density ρ_η, which maximizes $\mathcal{E}[\eta V]$ of (1.6), X_η of (1.24) corresponds to the support $[a_\eta, b_\eta]$ of ρ_η, and P_η is defined by (1.25) for ηV. The contour $\mathcal{L}$ here is chosen sufficiently close to $[a_\eta, b_\eta]$ to have z and all zeros of P_η outside of it. The remainder bound is uniform in $|\eta - 1| \le \varepsilon_1$. Moreover, for any φ with bounded fifth derivative $\varphi^{(5)}$

$$\begin{aligned} \int \varphi(\lambda)\Big(p_{1,\beta}^{(n,\eta)}(\lambda) - \rho_\eta(\lambda)\Big)d\lambda &= n^{-1}O(||\varphi||_\infty + ||\varphi^{(5)}||_\infty), \\ ||\varphi||_\infty &:= \sup_{\lambda\in\sigma_\varepsilon} |\varphi(\lambda)|. \end{aligned} \tag{1.29}$$

(ii) There exists an analytic in $\mathbf{D}\setminus\sigma_\varepsilon$ function u^* such that

$$u_{n,\eta}(z) - u_{n,1}(z) = (\eta - 1)u^*(z) + O((\eta-1)^2) + O(n^{-1}), \quad |\Im z| > d. \tag{1.30}$$

(iii) If $Q_{n,\beta}^{(\eta)}$ is defined by (1.2) for ηV, then

$$\begin{aligned} \log Q_{n,\beta}^{(\eta)} &= \log Q_{n,\beta}^* + \frac{\beta n^2}{2}\mathcal{E}[\eta V] + \frac{3\beta n^2}{8} + n\Big(1 - \frac{\beta}{2}\Big)\log\frac{d_\eta}{2} \\ &\quad -\frac{\beta n}{2}\int_0^1 dt \oint_{\mathcal{L}} (\eta V(z) - V_\eta^{(0)}(z))u_{n,\eta}(z,t)dz, \\ u_{n,\eta}(z,t) &= \frac{(2/\beta - 1)}{2\pi i X_\eta^{1/2}(z)}\oint_{\mathcal{L}_d}\frac{g'_\eta(\zeta,t)d\zeta}{P_\eta(\zeta,t)(z-\zeta)} + O(n^{-1}), \end{aligned} \tag{1.31}$$

where $\mathcal{E}[\eta V]$, X_η and $[a_\eta, b_\eta]$ are the same as in (ii),

$$\begin{aligned} &V_\eta^{(0)}(z) = 2(z - c_\eta)^2/d_\eta^2, \quad c_\eta = (a_\eta + b_\eta)/2, \quad d_\eta = (b_\eta - a_\eta)/2, \\ &P_\eta(\lambda, t) = tP_\eta(\lambda) + \frac{4(1-t)}{d_\eta^2}, \quad g_\eta(z,t) = tg_\eta(z) + \frac{2(1-t)}{d_\eta^2}(z - c_\eta - X_\eta^{1/2}(z)), \end{aligned}$$

$Q_{n,\beta}^*$ is defined by the Selberg formula

$$Q_{n,\beta}^* = n!\left(\frac{n\beta}{2}\right)^{-\beta n^2/4 - n(1-\beta/2)/2}(2\pi)^{n/2}\prod_{j=1}^n \frac{\Gamma(\beta j/2)}{\Gamma(\beta/2)}, \tag{1.32}$$

and the remainder bound is uniform in $|\eta - 1| \le \varepsilon_1$.

The next theorem establishes some important properties of the symplectic and orthogonal matrix models, in particular, it gives the bound for the rate of convergence of linear eigenvalue statistics and the bound for their variances. In order to formulate the theorem, we define for even n

$$\mu_\alpha^* = \int_{\sigma_\alpha}\rho(\lambda)d\lambda, \quad k_\alpha^* := [n\mu_\alpha^*] + d_\alpha, \tag{1.33}$$

where $[x]$ means an integer part of x, and we have chosen $d_\alpha = 0, \pm 1, \pm 2$ in a way which makes k_α^* even and

$$\sum k_\alpha^* = n. \tag{1.34}$$

For each $\sigma_{\alpha,\varepsilon}$ we introduce the "effective potential"

$$V_\alpha^{(a)}(\lambda) = \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\Big(V(\lambda) - 2\int_{\sigma\setminus\sigma_\alpha}\log|\lambda - \mu|\rho(\mu)d\mu\Big), \tag{1.35}$$

and denote Σ^* the "cross energy"

$$\Sigma^* : \ = \ \sum_{\alpha\neq\alpha'}\int_{\sigma_\alpha} d\lambda\int_{\sigma_{\alpha'}} d\mu\log|\lambda-\mu|\rho(\lambda)\rho(\mu). \tag{1.36}$$

Theorem 2 *If potential V satisfies conditions C1-C2 and n is even, then the matrices $F^{(1)}$ and $F^{(4)}$ in (1.18) are bounded in the operator norm uniformly in n. Moreover, for any smooth φ and $\beta = 1, 2, 4$ we have*

$$\Big|\int\varphi(\lambda)\Big(p_{1,\beta}^{(n)}(\lambda)-\rho(\lambda)\Big)d\lambda\Big| \le \frac{C}{n}||\varphi'||_\infty, \tag{1.37}$$
$$\mathbf{E}_\beta\left\{\Big|\mathcal{N}_n[\varphi]-\mathbf{E}_\beta\{\mathcal{N}_n[\varphi]\}\Big|^2\right\} \le C||\varphi'||_\infty^2,$$

where $||.||_\infty$ is defined in (1.29). The logarithm of the normalization constant $Q_{n,\beta}[V]$ can be obtained up to $O(1)$ term from the representation

$$\log(Q_{n,\beta}[V]/n!) \ = \ \sum_{\alpha=1}^{q}\log(Q_{k_\alpha^*,\beta}[V_\alpha^{(a)}]/k_\alpha^*!)-\frac{\beta n^2}{2}\Sigma^*+O(1), \tag{1.38}$$

where $V_\alpha^{(a)}$ and Σ^* are defined in (1.35) and (1.36).

As it was mentioned above, Theorem 2 together with some asymptotic results for orthogonal polynomials of [5] may be used to prove universality of local eigenvalue statistics of matrix models (1.1). In order to state our theorem on bulk universality, we need some more notations. Define

$$K_\infty(t) := \frac{\sin\pi t}{\pi t},$$
$$K_\infty^{(1)}(\xi,\eta) := \begin{pmatrix} K_\infty(\xi-\eta) & K_\infty'(\xi-\eta) \\ \int_0^{\xi-\eta}K_\infty(t)\,dt-\epsilon(\xi-\eta) & K_\infty(\eta-\xi)\end{pmatrix},$$
$$K_\infty^{(4)}(\xi,\eta) := \begin{pmatrix} K_\infty(\xi-\eta) & K_\infty'(\xi-\eta) \\ \int_0^{\xi-\eta}K_\infty(t)\,dt & K_\infty(\eta-\xi)\end{pmatrix}.$$

Furthermore, we denote for a 2×2 matrix A and $\lambda > 0$

$$A^{(\lambda)} := \begin{pmatrix}\sqrt{\lambda}^{-1} & 0 \\ 0 & \sqrt{\lambda}\end{pmatrix} A \begin{pmatrix}\sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1}\end{pmatrix}.$$

Theorem 3 *Let V satisfy conditions C1–C2. Then we have for (even) $n\to\infty$, $\lambda_0\in\mathbb{R}$ with $\rho(\lambda_0)>0$, and for $\beta\in\{1,4\}$ that*

$$q_n^{-1}K_{n,1}^{(q_n)}(\lambda_0+\xi/q_n,\lambda_0+\eta/q_n) = K_\infty^{(\beta)}(\xi,\eta)+O(n^{-1/2}),$$
$$q_n^{-1}K_{n/2,4}^{(q_n)}(\lambda_0+\xi/q_n,\lambda_0+\eta/q_n) = K_\infty^{(4)}(\xi,\eta)+O(n^{-1/2}),$$

where $q_n = n\rho(\lambda_0)$. The error bound is uniform for bounded ξ, η and for λ_0 contained in some compact subset of $\cup_{\alpha=1}^q(E_{2\alpha-1},E_{2\alpha})$.

It is an immediate consequence of Theorem 3 that the corresponding rescaled l-point correlation functions

$$p_{l,1}^{(n)}(\lambda_0+\xi_1/q_n,\ldots,\lambda_0+\xi_l/q_n), \quad p_{l,4}^{(n/2)}(\lambda_0+\xi_1/q_n,\ldots,\lambda_0+\xi_l/q_n)$$

converge for n (even) $\to\infty$ to some limit that depends on β but not on the choice of V.

The paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we prove Theorems 2 and 3 modulo some bounds which are obtained in Section 4.

2 Proof of Theorem 1

Proof of Theorem 1.
(i). Take n-independent ε, small enough to provide that $\sigma_\varepsilon \subset \mathbf{D}$. It is known (see e.g. [2]) that if we replace in (1.1), (1.4), and (1.5) the integration over $\mathbb{R}$ by the integration σ_ε, then the new marginal densities will differ from the initial ones by the terms $O(e^{-nc})$ with some c, depending on ε, but independent of n. Since for our purposes it is more convenient to consider the integration with respect to σ_ε, we assume from this moment that this replacement is made, so everywhere below the integration without limits means the integration over σ_ε.

Following the idea of [12], we will study a little bit modified form of the joint eigenvalue distribution, than in (1.1). Namely, consider some real smooth function $h(\lambda)$ and denote

$$V_{h,\eta}(\zeta) = \eta V(\zeta) + \frac{1}{n} h(\zeta). \tag{2.1}$$

Let $p_{n,\beta,h}$, $\mathbf{E}_{\beta,h}\{\dots\}$, $p_{l,\beta,h}^{(n,\eta)}$ be the distribution density, the expectation, and the marginal densities defined by (1.1), (1.5), and (1.4) with V replaced by $V_{h,\eta}$.

By (1.1) the first marginal density can be represented in the form

$$p_{1,\beta,h}^{(n,\eta)}(\lambda) = Q_{n,\beta,h}^{-1} \int e^{-n\beta V_{h,\eta}(\lambda)/2} \prod_{i=2}^{n} |\lambda - \lambda_i|^\beta e^{-n\beta V_{h,\eta}(\lambda_i)/2} \prod_{2\le i<j\le n} |\lambda_i - \lambda_j|^\beta d\lambda_2 \dots d\lambda_n. \tag{2.2}$$

Using the representation and integrating by parts, we obtain

$$\int \frac{V'_{h,\eta}(\lambda) p_{1,\beta,h}^{(n,\eta)}(\lambda)}{z-\lambda} d\lambda = \frac{2}{\beta n} \int \frac{p_{1,\beta,h}^{(n,\eta)}(\lambda)}{(z-\lambda)^2} d\lambda + \frac{2(n-1)}{n} \int \frac{p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} + O(e^{-nc}). \tag{2.3}$$

Here $O(e^{-nc})$ is the contribution of the integrated term. In fact all equations below should contain $O(e^{-nc})$, but in order to simplify formula below we omit it.

Since the function $p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu)$ is symmetric with respect to λ,μ, we have

$$2\int \frac{p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} = \int \frac{p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} + \int \frac{p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\mu)(\mu-\lambda)} = \int \frac{p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\lambda)(z-\mu)}.$$

Hence equation (2.3) can be written in the form

$$\int \frac{V'_{h,\eta}(\lambda) p_{1,\beta,h}^{(n,\eta)}(\lambda)}{z-\lambda} d\lambda = \frac{2}{\beta n} \int \frac{p_{1,\beta,h}^{(n,\eta)}(\lambda)}{(z-\lambda)^2} d\lambda + \frac{(n-1)}{n} \int \frac{p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\lambda)(z-\mu)}. \tag{2.4}$$

Let us introduce notations:

$$\begin{aligned} \delta_{\beta,h}^{(n,\eta)}(z) = n(n-1) \int \frac{p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\lambda)(z-\mu)} - n^2 \Big(\int \frac{p_{1,\beta,h}^{(n,\eta)}(\lambda) d\lambda}{z-\lambda} \Big)^2 + n \int \frac{p_{1,\beta,h}^{(n,\eta)}(\lambda)}{(z-\lambda)^2} d\lambda \\ = \int \frac{k_{\beta,h}^{(n,\eta)}(\lambda,\mu) d\lambda d\mu}{(z-\lambda)(z-\mu)}, \end{aligned} \tag{2.5}$$

where

$$k_{\beta,h}^{(n,\eta)}(\lambda,\mu) = n(n-1) p_{2,\beta,h}^{(n,\eta)}(\lambda,\mu) - n^2 p_{1,\beta,h}^{(n,\eta)}(\lambda) p_{1,\beta,h}^{(n,\eta)}(\mu) + n\delta(\lambda-\mu) p_{1,\beta,h}^{(n,\eta)}(\lambda). \tag{2.6}$$

Moreover, we denote

$$g^{(\eta)}_{n,\beta,h}(z)=\int\frac{p^{(n,\eta)}_{1,\beta,h}(\lambda)d\lambda}{\lambda-z},\quad V(z,\lambda)=\frac{V'(z)-V'(\lambda)}{z-\lambda}. \tag{2.7}$$

Then equation (2.3) takes the form

$$\begin{aligned}(g^{(\eta)}_{n,\beta,h}(z))^2&+\eta V'(z)g^{(\eta)}_{n,\beta,h}(z)+\eta\int V(z,\lambda)p^{(n,\eta)}_{1,\beta,h}(\lambda)d\lambda\\ &=\frac{1}{n}\int\frac{h'(\lambda)p^{(n,\eta)}_{1,\beta,h}(\lambda)}{z-\lambda}d\lambda-\frac{1}{n}\Big(\frac{2}{\beta}-1\Big)\int\frac{p^{(n,\eta)}_{1,\beta,h}(\lambda)}{(z-\lambda)^2}d\lambda-\frac{1}{n^2}\delta_{n,\beta,h}(z).\end{aligned} \tag{2.8}$$

Using that $V(z,\zeta)$ is an analytic functions of ζ in $\mathbf{D}$, by the Cauchy theorem we obtain that

$$\int V(z,\lambda)p^{(n,\eta)}_{1,\beta,h}(\lambda)d\lambda=-\frac{1}{2\pi i}\oint_{\mathcal{L}}V(z,\zeta)g^{(\eta)}_{n,\beta,h}d\zeta.$$

if $\mathcal{L}$ is encircling σ_ε, and z is outside of $\mathcal{L}$. Thus (2.8) takes the form

$$\begin{aligned}(g^{(\eta)}_{n,\beta,h}(z))^2&+\eta V'(z)g_{n,\beta,h}(z)-\frac{\eta}{2\pi i}\oint_{\mathcal{L}}V(z,\zeta)g^{(\eta)}_{n,\beta,h}(\zeta)d\zeta\\ &=\frac{1}{n}\int\frac{h'(\lambda)p^{(n,\eta)}_{1,\beta,h}(\lambda)}{z-\lambda}d\lambda-\frac{1}{n}\Big(\frac{2}{\beta}-1\Big)\int\frac{p^{(n,\eta)}_{1,\beta,h}(\lambda)}{(z-\lambda)^2}d\lambda-\frac{1}{n^2}\delta_{n,\beta,h}(z).\end{aligned} \tag{2.9}$$

On the other hand, it is easy to show that g_η satisfies the equation

$$g^2_\eta(z)+V'(z)g_\eta(z)+Q_\eta(z)=0,\quad Q(z)=-\frac{\eta}{2\pi i}\oint_{\mathcal{L}}V(z,\zeta)g_\eta(\zeta)d\zeta. \tag{2.10}$$

Hence

$$g_\eta(z)=-\frac{\eta}{2}V'(z)+\frac{1}{2}\sqrt{\eta^2V'(z)^2-4Q_\eta(z)}.$$

Using the inverse Stieltjes transform and comparing with (1.23), we get that

$$2g_\eta(z)+\eta V'(z)=P_\eta(z)X^{1/2}_\eta(z). \tag{2.11}$$

where $X_\eta(z)$ is defined by (1.24) for q=1.

Write $g^{(\eta)}_{n,\beta,h}=g_\eta+n^{-1}u_{n,\eta}$. Then, subtracting (2.10) from (2.9) and multiplying the result by n, we get

$$(2g_\eta(z)+\eta V'(z))u_{n,\eta}(z)-\frac{\eta}{2\pi i}\oint_{\mathcal{L}}V(z,\zeta)u_{n,\eta}(\zeta)d\zeta=F(z), \tag{2.12}$$

where

$$\begin{aligned}F(z)\quad=&\int\frac{h'(\lambda)p^{(n,\eta)}_{1,\beta,h}(\lambda)}{z-\lambda}d\lambda-\Big(\frac{2}{\beta}-1\Big)\Big(g'_\eta(z)+\frac{1}{n}u'_{n,\eta}(z)\Big)\\ &-\frac{1}{n}u^2_{n,\eta}(z)-\frac{1}{n}\delta^{(n,\eta)}_{\beta,h}(z).\end{aligned} \tag{2.13}$$

Using (2.11), we obtain from (2.12)

$$P_\eta(z)X^{1/2}_\eta(z)u_{n,\eta}(z)+\mathcal{Q}_n(z)=F(z),\quad \mathcal{Q}_n(z)=-\frac{\eta}{2\pi i}\oint V(z,\zeta)u_{n,\eta}(\zeta)d\zeta. \tag{2.14}$$

Then, choosing $\mathcal{L}$ sufficiently close to $[a_\eta, b_\eta]$ to have all zeros of $P(\zeta)$ and z outside of $\mathcal{L}$, we get

$$\frac{1}{2\pi i}\oint_{\mathcal{L}}\Big(P_\eta(\zeta)X_\eta^{1/2}(\zeta)u_{n,\beta,h}(\zeta)+\mathcal{Q}_n(\zeta)-F(\zeta)\Big)\frac{d\zeta}{P_\eta(\zeta)(z-\zeta)}=0. \tag{2.15}$$

Since by definition (2.14) $\mathcal{Q}_n(\zeta)$ is analytic in $\mathbf{D}$, and z and all zeros of P are outside of $\mathcal{L}$, the Cauchy theorem yields

$$\frac{1}{2\pi i}\oint_{\mathcal{L}}\frac{\mathcal{Q}_n(\zeta)d\zeta}{P_\eta(\zeta)(z-\zeta)}=0.$$

Moreover, since

$$u_{n,\eta}(z)=\frac{n}{z}\left(\int p_{1,\beta,h}^{(n,\eta)}(\lambda)d\lambda-\int\rho_\eta(\lambda)d\lambda\right)+nO(z^{-2})=nO(z^{-2}),\quad z\to\infty,$$

we have

$$X_\eta^{1/2}(z)u_{n,\eta}(z)=nO(z^{-1}). \tag{2.16}$$

Then the Cauchy theorem yields

$$\frac{1}{2\pi i}\oint_{\mathcal{L}}\frac{X_\eta^{1/2}(\zeta)u_{n,\eta}(\zeta)d\zeta}{(z-\zeta)}=X_\eta^{1/2}(z)u_{n,\eta}(z).$$

Finally, we obtain from (2.15)

$$u_{n,\eta}(z)=\frac{1}{2\pi iX_\eta^{1/2}(z)}\oint_{\mathcal{L}}\frac{F(\zeta)d\zeta}{P_\eta(\zeta)(z-\zeta)}. \tag{2.17}$$

Set

$$d(z)=\operatorname{dist}\{z,\sigma_\varepsilon\}, \tag{2.18}$$

Then for any $z: d(z)=d$ equation (2.17) implies

$$u_{n,\eta}(z)=\frac{F(z)}{X_\eta^{1/2}(z)P_\eta(z)}+\frac{1}{2\pi iX_\eta^{1/2}(z)}\oint_{\mathcal{L}'}\frac{F(\zeta)d\zeta}{P_\eta(\zeta)(\zeta-z)}, \tag{2.19}$$

where the contour $\mathcal{L}'$ contains z but does not contain zeros of $P(z)$. It will be convenient for us to take $\mathcal{L}'$ as far from σ_ε, as it is possible. According to [2], for any β and η we have the bounds:

$$|\delta_{\beta,h}^{(n,\eta)}(z)|\le\frac{Cn\log n}{d^4(z)},\qquad |u_{n,\eta}(z)|\le\frac{Cn^{1/2}\log^{1/2}n}{d^2(z)},\quad |u_{n,\eta}'(z)|\le\frac{Cn^{1/2}\log^{1/2}n}{d^3(z)}, \tag{2.20}$$

where C is an absolute constant. Set

$$M_n(d)=\sup_{z:d(z)\ge d}|u_{n,\eta}(z)|.$$

By (2.16) and the maximum principle, there exists a point $z: d(z)=d$ such that

$$M_n(d)=|u_{n,\eta}(z)|.$$

Then, using (2.19), the definition of F (2.13), and (2.20), we obtain the inequality

$$M_n(d)\le\frac{M_n^2(d)}{C_1nd}+\frac{C_2\log n}{d^5},\quad C_2\le C_0(1+||h'||_\infty)$$

with some C_0, C_1 depending only on P and C of (2.20). Solving the above quadratic inequality, we get

$$\left[\begin{array}{l} M_n(d) \ge \frac{1}{2}\Big(C_1 nd + \sqrt{C_1^2 n^2 d^2 - 4C_1C_2 n\log n/d^4}\Big); \\ M_n(d) \le \frac{1}{2}\Big(C_1 nd - \sqrt{C_1^2 n^2 d^2 - 4C_1C_2 n\log n/d^4}\Big). \end{array}\right.$$

Since the first inequality contradicts to (2.20), we conclude that for $d > n^{-1/6}\log n$ the second inequality holds. Hence we get

$$|u_{n,\eta}(z)| \le C_0 \log n d^{-5}(z)(1 + ||h'||_\infty), \quad d(z) > n^{-1/6}\log n. \tag{2.21}$$

Note that the bound gives us that if we consider $\varphi(\lambda) = \Re(\lambda - z)^{-1}$ or $\varphi(\lambda) = \Im(\lambda - z)^{-1}$ with $d(z) > n^{-1/6}\log n$, then

$$n\left|\int \varphi(\lambda)(p_{1,\beta,h}^{(n,\eta)}(\lambda) - \rho_\eta(\lambda))d\lambda\right| \le w_n||\varphi^{(s)}||_\infty(1 + ||h'||_\infty), \tag{2.22}$$

where $\varphi^{(s)}$ is the s-th derivative of φ (now we have $s = 4$ in view of (2.21)) and

$$w_n = sC_0 \log n$$

We are going to use the following lemma, which is an analog of Lemma 3.11 of [12].

Lemma 1 *If (2.22) holds for any real h : $||h'||_\infty < A$ ($A > 1$) and some φ such that $||\varphi^{(s)}||_\infty \ge ||\varphi'||_\infty$, then there exists an absolute constant C_* such that for any $||h'||_\infty < A/2$*

$$\int k_{\beta,h}^{(n,\eta)}(\lambda,\mu)\varphi(\lambda)\varphi(\mu)d\lambda d\mu \le C_* w_n^2(1 + A)^2||\varphi^{(s)}||_\infty^2 \tag{2.23}$$

The lemma was proven in [12], but for reader's convenience we give its proof here.

Proof of Lemma 1. Without loss of generality assume that $w_n > 1$. Using the method of [12], consider the function

$$Z_n(t) = \mathbf{E}_{\eta,\beta,h}\left\{\exp\left[\frac{t}{2\widetilde{w}_n}\sum_{i=1}^n\Big(\varphi(\lambda_i) - \int\varphi(\lambda)\rho_\eta(\lambda)d\lambda\Big)\right]\right\}, \quad \widetilde{w}_n = w_n(1 + A)||\varphi^{(s)}||_\infty,$$

where $\mathbf{E}_{\eta,\beta,h}\{.\}$ is defined in (1.5) with V replaced by $V_{\eta,h}$ of (2.1). It is easy to see that

$$\frac{d^2}{dt^2}\log Z_n(t) = (2w_n)^{-2}\mathbf{E}_{\beta,h+t\varphi/2\widetilde{w}_n}\left\{\left(\sum_{i=1}^n(\varphi(\lambda_i) - \mathbf{E}_{\beta,h+t\varphi/2\widetilde{w}_n}\{\varphi(\lambda_i)\})\right)^2\right\} \ge 0. \tag{2.24}$$

Hence in view of (2.22),

$$\begin{aligned} \log Z_n(t) = \log Z_n(t) - \log Z_n(0) &= \int_0^t \frac{d}{d\tau}\log Z_n(\tau)d\tau \le |t|\frac{d}{dt}\log Z_n(t) \\ &= |t|(2\widetilde{w}_n)^{-1}\mathbf{E}_{\eta,\beta,h+t\varphi/2\widetilde{w}_n}\left\{\sum_{i=1}^n\Big(\varphi(\lambda_i) - \int\varphi(\lambda)\rho_\eta(\lambda)d\lambda\Big)\right\} \\ &= \frac{|t|n}{2\widetilde{w}_n}\int\varphi(\lambda)\Big(p_{1,\beta,h+t\varphi/2\widetilde{w}_n}^{(n,\eta)}(\lambda) - \rho_\eta(\lambda)\Big)d\lambda \le |t|, \qquad t \in [-1,1]. \end{aligned}$$

Thus

$$Z_n(t) \le e^{|t|} \le 3, \quad t \in [-1,1],$$

and for any $t \in \mathbb{C}$, $|t| \le 1$

$$|Z_n(t)| \le Z_n(\Re t) < 3. \tag{2.25}$$

Then, by the Cauchy theorem, we have

$$|Z_n'(t)| = \left| \frac{1}{2\pi} \oint_{|t'|=1} \frac{Z_n(t')dt'}{(t'-t)^2} \right| \le 12, \quad |t| \le \frac{1}{2},$$

and therefore for $|t| \le \frac{1}{24}$

$$|Z_n(t)| = |Z_n(0) - \int_0^t Z_n'(t)dt| \ge \frac{1}{2}.$$

Hence $\log Z_n(t)$ is analytic for $|t| \le \frac{1}{24}$, and using the above bounds we have

$$\frac{d^2}{dt^2} \log Z_n(0) = \frac{1}{2\pi i} \oint_{|t|=1/24} \frac{\log Z_n(t)}{t^3} dt \le C.$$

Finally, in view of (2.24) we get

$$\int k_{n,\beta,h}(\lambda,\mu)\varphi(\lambda)\varphi(\mu)d\lambda d\mu = \mathbf{E}_{\eta,\beta,h}\left\{ \left(\sum_{i=1}^n (\varphi(\lambda_i) - \mathbf{E}_{\eta,\beta,h}\{\varphi(\lambda_i)\}) \right)^2 \right\} \le 4Cw_n^2.$$

□

Let us come back to the proof of Theorem 1. Applying the lemma to $\varphi_z^{(1)}(\lambda) = \Re(z-\lambda)^{-1}$ and $\varphi_z^{(2)}(\lambda) = \Im(z-\lambda)^{-1}$ with $d(z) > n^{-1/6}\log n$ and using (2.22) for $||h'||_\infty \le 2$, we obtain for such z (cf (2.20))

$$|\delta_{\beta,h}^{(n,\eta)}(z)| \le C \log^2 n d^{-10}(z) \tag{2.26}$$

Then, using this bound and (2.21) in (2.17) and taking into account that $n^{-1}\delta_{\beta,h}^{(n,\eta)}(z) \le d^{-4}(z)$ for $d(z) > n^{-1/6}\log n$, we obtain that for $||h'||_\infty \le 1$

$$u_{n,\eta}(z) = \frac{(2/\beta - 1)}{2\pi i X_\eta^{1/2}(z)} \oint_{\mathcal{L}_d} \frac{g_\eta'(\zeta)d\zeta}{P_\eta(\zeta)(z-\zeta)} + O(d^{-5}(z)). \tag{2.27}$$

Applying Lemma 1 once more, we get $\delta_{\beta,h}^{(n,\eta)}(z) \le Cd^{-10}(z)$. Using the bound in (2.19) we prove (1.28).

To prove (1.29) consider the Poisson kernel

$$\mathcal{P}_y(\lambda) = \frac{y}{\pi(y^2+\lambda^2)}. \tag{2.28}$$

It is easy to see that for any integrable φ

$$(\mathcal{P}_y * \varphi)(\lambda) = \frac{1}{\pi}\Im \int \frac{\varphi(\mu)d\mu}{\mu - (\lambda + iy)}.$$

Hence we can use (2.27) to prove that for $|y| \ge n^{-1/6}\log n$

$$||\mathcal{P}_y * u_{n,\eta}||_2^2 \le Cy^{-10}, \quad \text{for} \quad u_{n,\eta}(\lambda) := n(p_{1,\beta}^{(n,\eta)}(\lambda) - \rho_\eta(\lambda)), \tag{2.29}$$

where $||.||_2$ is the standard norm in $L_2(\mathbb{R})$. Then we use the formula (see [12]) valid for any $u \in L_2(\mathbb{R})$

$$\int_0^\infty e^{-y} y^{2s-1} ||\mathcal{P}_y * u||_2^2 dy = \Gamma(2s) \int_{\mathbb{R}} (1+2|\xi|)^{-2s} |\widehat{u}(\xi)|^2 d\xi. \tag{2.30}$$

The formula for $s = 5$, the Parseval equation for the Fourier integral, and the Schwarz inequality yield

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(\lambda) u_{n,\eta}(\lambda) d\lambda &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) \widehat{u}_{n,\eta}(\xi) d\xi \\
&\le \frac{1}{2\pi} \left(\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 (1+2|\xi|)^{2s} d\xi \right)^{1/2} \left(\int_{\mathbb{R}} |\widehat{u}_{n,\eta}(\xi)|^2 (1+2|\xi|)^{-2s} d\xi \right)^{1/2} \\
&\le \frac{C((||\varphi||_2 + ||\varphi^{(5)}||_2)}{\Gamma^{1/2}(2s)} \left(\int_0^\infty e^{-y} y^{2s-1} ||P_y * u_{n,\eta}||_2^2 dy \right)^{1/2}.
\end{aligned}$$

To estimate the last integral here we split it into two parts $|y| \ge n^{-1/6} \log n$ and $|y| < n^{-1/6} \log n$. For the first integral we use (2.29) and for the second - (2.20).

(ii) To prove (1.30) we start from the relations

$$\begin{aligned}
&a_\eta - a = (\eta - 1) a_* + O((\eta-1)^2), \quad b_\eta - b = (\eta-1) b_* + O((\eta-1)^2), \\
&P_\eta(z) - P(z) = (\eta - 1) p(z) + O((\eta-1)^2), \\
&g_\eta(z) - g(z) = (\eta - 1) m(z) + O((\eta-1)^2 |\Im z|^{-3}),
\end{aligned} \tag{2.31}$$

where a_*, b_* are some constant, $p(z)$ is some analytic in $\mathbf{D}$ function, and $m(z)$ is analytic outside $[a, b]$. The first line here follows from the results of [10]. The second line follows from the first one, if we use the representation (1.25). The third line follows from the representation

$$g_\eta(z) = \frac{1}{2} (X_\eta^{1/2}(z) P_\eta(z) - \eta V'(z))$$

and the first two lines. Then (1.30) follows from (1.28), combined with (2.31).

(iii) Consider the functions $V_{\eta,t}$ of the form

$$V_{\eta,t}(\lambda) = t\eta V(\lambda) + (1-t) V_\eta^{(0)}(\lambda), \tag{2.32}$$

where $V_\eta^{(0)}$ is defined in (1.31). Let $Q_{n,\beta}^{(\eta)}(t) := Q_{n,\beta}[V_{\eta,t}]$ be defined by (1.2) with V replaced by $V_{\eta,t}$. Then, evidently, $Q_{n,\beta}^{(\eta)}(1) = Q_{n,\beta}[\eta V]$, and $Q_{n,\beta}(0)$ corresponds to $V_\eta^{(0)}$ (see (1.31)). Hence

$$\begin{aligned}
\frac{1}{n^2} \log Q_{n,\beta}^{(\eta)}(1) - \frac{1}{n^2} \log Q_{n,\beta}^{(\eta)}(0) &= \frac{1}{n^2} \int_0^1 dt \frac{d}{dt} \log Q_{n,\beta}^{(\eta)}(t) \\
&= -\frac{\beta}{2} \int_0^1 dt \int d\lambda (\eta V(\lambda) - V_\eta^{(0)}(\lambda)) p_{1,\beta}^{(n,\eta)}(\lambda; t),
\end{aligned} \tag{2.33}$$

where $p_{1,\beta}^{(n,\eta)}(\lambda; t)$ is the first marginal density, corresponding to $V_{\eta,t}$. Using (1.8), one can check that for the distribution (1.1) with V replaced by V_t the equilibrium density ρ_t has the form

$$\rho_t(\lambda) = t\rho_\eta(\lambda) + (1-t)\rho_\eta^{(0)}(\lambda), \quad \rho_\eta^{(0)}(\lambda) = \frac{2X_\eta(\lambda)}{\pi d_\eta^2} \tag{2.34}$$

with X_η, d_η of (1.31). Hence using (1.28) for the last integral in (2.33), we get

$$\begin{aligned}\log Q_{n,\beta}[\eta V] &= \log Q_{n,\beta}[V_\eta^{(0)}] - n^2\frac{\beta}{2}\mathcal{E}[V_\eta^{(0)}] + n^2\frac{\beta}{2}\mathcal{E}[\eta V]\\ &\quad -\frac{\beta n}{2}\frac{1}{(2\pi i)}\int_0^1 dt \oint_{\mathcal{L}_{2d}} (\eta V(z) - V_\eta^{(0)}(z))u_{n,\eta}(z,t)dz,\end{aligned}$$

Changing the variables in the corresponding integrals, we have

$$\begin{aligned}\log Q_{n,\beta}[V_\eta^{(0)}] &= \log Q^*_{n,\beta} + \Big(\frac{n^2\beta}{2} + n(1-\beta/2)\Big)\log\frac{d_\eta}{2},\\ \frac{n^2\beta}{2}\mathcal{E}[V_\eta^{(0)}] &= -\frac{3n^2\beta}{8} + \frac{n^2\beta}{2}\log\frac{d_\eta}{2}.\end{aligned}$$

Then (1.31) follows. □

3 Matrix kernels for orthogonal and symplectic ensembles

The orthonormal system $\{\psi_k^{(n)}\}_{k=0}^\infty$ satisfies the recursion relations

$$\lambda\psi_k^{(n)}(\lambda) = a_{k+1}^{(n)}\psi_{k+1}^{(n)}(\lambda) + b_k^{(n)}\psi_k^{(n)}(\lambda) + a_k^{(n)}\psi_{k-1}^{(n)}(\lambda), \tag{3.1}$$

which define a semi-infinite Jacobi matrix $J^{(n)}$. It is known (see, e.g. [16]) that

$$|a_k^{(n)}| \le C, \quad |b_k^{(n)}| \le C, \quad |n-k| \le \varepsilon n. \tag{3.2}$$

By orthogonality and the spectral theorem, we see

$$\begin{aligned}&(D_\infty^{(n)})_{j,k} = n\,\mathrm{sign}(j-k)V'(J^{(n)})_{jk} \Rightarrow \\ \Rightarrow\quad &(D_\infty^{(n)})_{j,k} = 0,\ |j-k|\ge 2m, \quad |(D_\infty^{(n)})_{j,k}| \le nC,\ \ |j-n|\le nc.\end{aligned} \tag{3.3}$$

We are going to use the formula for $S_{n,\beta}$, obtained in [23] (see also [8]). In order to present this formula, introduce some more notations:

$$\begin{aligned}\Phi_1^{(n)} &:= (\psi_{n-2m+1}^{(n)}, \psi_{n-2m+2}^{(n)}, \dots, \psi_{n-1}^{(n)})^T,\\ \Phi_2^{(n)} &:= (\psi_n^{(n)}, \psi_{n+1}^{(n)}, \dots, \psi_{n+2m-2}^{(n)})^T,\end{aligned}$$

and

$$M_{rs} := (\epsilon\Phi_r^{(n)}, (\Phi_s^{(n)})^T), \qquad D_{rs} := ((\Phi_r^{(n)})', (\Phi_s^{(n)})^T), \quad 1\le r,s\le 2.$$

Observe that $M_\infty^{(n)}D_\infty^{(n)} = 1$ together with $(D_\infty^{(n)})_{jk} = 0$ for $|j-k|\ge 2m$ implies

$$T_n = 1 - D_{12}M_{21}.$$

Then have (see [8])

$$\begin{aligned}S_{n,1}(\lambda,\mu) &= K_{n,2}(\lambda,\mu) + \Phi_1(\lambda)^T D_{12}\epsilon\Phi_2(\mu) - \Phi_1(\lambda)^T\hat{G}\epsilon\Phi_1(\mu),\\ \hat{G} &:= D_{12}M_{22}(1-D_{21}M_{12})^{-1}D_{21}\\ S_{n/2,4}(\lambda,\mu) &= K_{n,2}(\lambda,\mu) + \Phi_2(\lambda)^T D_{12}\epsilon\Phi_1(\mu) - \Phi_2(\lambda)^T G\epsilon\Phi_2(\mu),\\ G &:= -D_{21}(1-M_{12}D_{21})^{-1}M_{11}D_{12},\end{aligned} \tag{3.4}$$

where $K_{n,2}$ is defined in (1.10).

Proof of Theorem 2. Using (3.4), it is straightforward to see that the first assertion of Theorem 2 follows from the following lemma.

Lemma 2 *Given any smooth functions f, g and any fixed $A > 0$ there exists a $C > 0$ such that for all $n \geq 2m$ and all $j, k \in \{n - A, \ldots, n + A\}$ one has*

$$(i) \quad \Big| \int \epsilon(f\psi_j^{(n)})(\lambda) g(\lambda)\psi_k^{(n)})(\lambda) d\lambda \Big| \leq \frac{C}{n}\Big(||f||_\infty + ||f'||_\infty\Big)\Big(||g||_\infty + ||g'||_\infty\Big); \quad (3.5)$$

$$(ii) \quad \epsilon(f\psi_j^{(n)})(\lambda)| \leq \frac{C}{\sqrt{n}}\Big(||f||_\infty + ||f'||_\infty\Big); \quad (iii) \quad \log\det(T_n) \geq C.$$

Indeed, taking in (i) $f = g = 1$, we obtain that all entries of $M_\infty^{(n)}$ which are used in (3.4) are bounded by Cn^{-1}, hence, having that $|\det T|^{-1} \leq C$ we obtain that G and $\hat{G}$ have entries bounded by nC. This proves representation (1.18).

Proof of Lemma 2. Assertions (i) and (ii) of Lemma 2 will be derived from the asymptotics of the orthogonal polynomials in Section 4. We now prove assertion (iii), using Theorem 1.

Note that without lost of generality we can assume that $\sigma \subset (-1, 1)$ and $v^* = 0$ in (1.8).

Similarly to the proof of Theorem 1 we choose ε in (1.26) small enough to have all zeros of P of (1.25) outside of σ_ε and use the results of [17] that if we replace in all definition (1.1) – (1.4) the integration in $\mathbb{R}$ by integration with respect to σ_ε, then the new $Q_{n,\beta}^{(\varepsilon)}[V]$ will differ from $Q_{n,\beta}[V]$ by the factor $(1+O(e^{-nc}))$, where $c > 0$ does not depend on n, but depends on ε. Moreover, the new marginal densities will differ from (1.4) by an additive error $O(e^{-nc})$. Hence starting from now, we assume that this replacement is made and all integrals below are in σ_ε.

Consider the "approximating" function H_a (Hamiltonian) (cf (1.1))

$$\begin{aligned} H_a(\lambda_1 \ldots \lambda_n) &= -n\sum V^{(a)}(\lambda_i) + \sum_{i \neq j} \log|\lambda_i - \lambda_j| \Big(\sum_{\alpha=1}^{q} \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda_i)\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda_j)\Big) - n^2\Sigma^*, \\ V^{(a)}(\lambda) &= \sum_{\alpha=1}^{q} V_\alpha^{(a)}(\lambda), \end{aligned} \quad (3.6)$$

where $V_\alpha^{(a)}(\lambda)$ is defined in (1.35), and Σ^* is defined in (1.36). Then

$$\begin{aligned} H(\lambda_1 \ldots \lambda_n) &= H_a(\lambda_1 \ldots \lambda_n) + \Delta H(\lambda_1 \ldots \lambda_n), \quad \lambda_1, \ldots, \lambda_n \in \sigma_\varepsilon, \quad (3.7) \\ \Delta H(\lambda_1 \ldots \lambda_n) &= \sum_{i \neq j} \log|\lambda_i - \lambda_j| \sum_{\alpha \neq \alpha'} \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda_i)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\lambda_j) - 2n\sum_{j=1}^{n} \widetilde{V}(\lambda_j) + n^2\Sigma^*, \\ \widetilde{V}(\lambda) &= \sum_{\alpha=1}^{q} \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda) \int_{\sigma \setminus \sigma_\alpha} \log|\lambda - \mu|\rho(\mu) d\mu. \end{aligned}$$

Set

$$Q_{n,\beta}^{(a)} = \int_{\sigma_\varepsilon^n} e^{\beta H_a(\lambda_1 \ldots \lambda_n)} d\lambda_1 \ldots d\lambda_n. \quad (3.8)$$

By the Jensen inequality, we have

$$\beta \langle \Delta H \rangle_{H^a,\beta} \leq \log Q_{n,\beta}[V] - \log Q_{n,\beta}^{(a)} \leq \beta \langle \Delta H \rangle_{H,\beta}, \quad (3.9)$$

where

$$< \cdots >_{H,\beta} := Q_{n,\beta}^{-1} \int (\ldots) e^{\beta H(\bar{\lambda})} d\bar{\lambda}, \quad < \cdots >_{H_a,\beta} := (Q_{n,\beta}^{(a)})^{-1} \int (\ldots) e^{\beta H_a(\bar{\lambda})} d\bar{\lambda}.$$

Let us estimate the r.h.s. of (3.9) for $\beta = 2$.

$$\langle \Delta H\rangle_{H,\beta} = n(n-1)\int p_{2,\beta}^{(n)}(\lambda,\mu)\log|\lambda-\mu|\sum_{\alpha\neq\alpha'}^{q}\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\mu) - 2n^2\big(\widetilde{V}, p_{1,\beta}^{(n)}\big) + n^2L^*.$$

Here and below $(.,.)$ means the inner product in $L_2(\sigma_\varepsilon)$. But using the definition of $\widetilde{V}$ and L^* we can rewrite the r.h.s. above as

$$\langle \Delta H\rangle_{H,\beta} = \sum_{\alpha\neq\alpha'}\int \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\mu)\log|\lambda-\mu|k_\beta^{(n)}(\lambda,\mu)d\lambda d\mu \qquad (3.10)$$

$$+n^2\sum_{\alpha\neq\alpha'}L\Big[\mathbf{1}_{\sigma_{\alpha,\varepsilon}}\big(p_{1,\beta}^{(n)}-\rho\big),\mathbf{1}_{\sigma_{\alpha',\varepsilon}}\big(p_{1,\beta}^{(n)}-\rho\big)\Big]\le C,$$

where the kernel $k_\beta^{(n)}$ is defined in (2.6) with $\eta = 1$ and $h = 0$. The term with $\delta(\lambda-\mu)$ from (2.6) gives zero contribution here, because in our integration domain $|\lambda-\mu|\ge\delta$ (see (1.26)). The last inequality in (3.10) is obtained as follows. Since $\mathrm{dist}\,\{\sigma_{\alpha,\varepsilon},\sigma_{\alpha',\varepsilon}\}\ge\delta$, and $\sigma_\varepsilon\subset[-1,1]$, we can construct 6-periodic even function $\widetilde{L}_{\alpha,\alpha'}(\lambda)$ with 12 derivatives such that

$$\widetilde{L}_{\alpha,\alpha'}(\lambda-\mu) = \log|\lambda-\mu|,\ \lambda\in\sigma_{\alpha,\varepsilon},\ \mu\in\sigma_{\alpha',\varepsilon},\quad \widetilde{L}_{\alpha,\alpha'}(\lambda) = 0,\ |\lambda|>5/2. \qquad (3.11)$$

Then,

$$L_{\alpha,\alpha'}(\lambda-\mu) = \sum_{k=-\infty}^{\infty}c_ke^{ik\pi(\lambda-\mu)/3},\quad \lambda\in\sigma_{\alpha,\varepsilon},\ \mu\in\sigma_{\alpha',\varepsilon},\quad |c_k|\le Ck^{-12}. \qquad (3.12)$$

Note that we need 12 derivatives to have for the Fourier coefficients of $L_{\alpha,\alpha'}$ the above bound, which will be used later. Hence

$$\int \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\mu)\log|\lambda-\mu|k_\beta^{(n)}(\lambda,\mu)d\lambda d\mu$$

$$= \sum_{k=-\infty}^{\infty}c_k\int \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(\mu)e^{ik\pi\lambda/3}e^{-ik\pi\mu/3}k_\beta^{(n)}(\lambda,\mu)d\lambda d\mu = O(1).$$

Here we used the well known bound for $\beta = 2$ (see e.g. [17])

$$\int\varphi(\lambda)\overline{\varphi(\mu)}k_\beta^{(n)}(\lambda,\mu)d\lambda d\mu = \int|\varphi(\lambda)-\varphi(\mu)|^2K_{n,2}^2(\lambda,\mu)d\lambda d\mu\le C||\varphi'(\lambda)||_\infty. \qquad (3.13)$$

For each term of the second sum in (1.28) we have similarly

$$L\big[\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(p_{1,\beta}^{(n)}-\rho),\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(p_{1,\beta}^{(n)}-\rho)\big] \qquad (3.14)$$

$$= \sum_{k=-\infty}^{\infty}c_k\left(\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(p_{1,\beta}^{(n)}-\rho),e^{ik\pi\lambda/2}\right)\left(\mathbf{1}_{\sigma_{\alpha',\varepsilon}}(p_{1,\beta}^{(n)}-\rho),e^{-ik\pi\lambda/2}\right) = O(n^{-2}).$$

This follows from the results of [5], according to which for any smooth φ

$$\int\varphi(\lambda)\Big(p_{1,2}^{(n)}(\lambda)-\rho(\lambda)\Big)d\lambda = O(n^{-1})||\varphi'||_\infty. \qquad (3.15)$$

To estimate the l.h.s. of (3.9), we study first the structure of $Q^{(a)}_{n,\beta}$. Since H_a does not contain terms $\log|\lambda_i-\lambda_j|$ with $\lambda_i\in\sigma_{\alpha,\varepsilon}$, $\lambda_j\in\sigma_{\alpha',\varepsilon}$ with $\alpha\neq\alpha'$, it can be written as

$$\begin{aligned}
Q^{(a)}_{n,\beta} &= \sum_{k_1+\cdots+k_q=n}\frac{n!}{k_1!\dots k_q!}Q_{\bar k,\beta}[V^{(a)}],\\
Q_{\bar k,\beta} &= \int_{\sigma_\varepsilon^n}\prod_{j=1}^{k_1}\mathbf{1}_{\sigma_{1,\varepsilon}}(\lambda_j)\cdots\prod_{j=k_1+\cdots+k_{q-1}+1}^{n}\mathbf{1}_{\sigma_{q,\varepsilon}}(\lambda_j)e^{\beta H_a(\lambda_1\dots\lambda_n)}d\lambda_1\dots d\lambda_n \qquad (3.16)\\
&= e^{-n^2\beta\Sigma^*/2}\prod_{\alpha=1}^{q}Q_{k_\alpha,\beta}[nV_a^{(\alpha)}/k_\alpha],
\end{aligned}$$

where the "effective potentials" $\{V_a^{(\alpha)}\}_{\alpha=1}^q$ are defined in (1.35). Take $\{k^*_\alpha\}_{\alpha=1}^q$ of (1.33) and set

$$\kappa_{\bar k}:=\frac{k_1^*!\dots k_q^*!}{k_1!\dots k_q!}Q_{\bar k;\beta}/Q_{\bar k^*;\beta}. \qquad (3.17)$$

It is evident that

$$Q^{(a)}_{n,\beta} = \frac{n!}{k_1^*!\dots k_q^*!}Q_{\bar k^*;\beta}\sum_{k_1+\cdots+k_q=n}\kappa_{\bar k}.$$

Choose $\widetilde\varepsilon$ sufficiently small to provide that $|\mu^*_\alpha n/k_\alpha-1|\le\varepsilon_1^{(\alpha)}$ for $\alpha=1,\dots,q$, if $|\bar k-\bar k^*|\le\widetilde\varepsilon n$, where $\varepsilon_1^{(\alpha)}$ is chosen for $\sigma_{\alpha,\varepsilon}$ and $(n/k_\alpha)V_a^{(\alpha)}$, according to Theorem 1.

Let us prove that there exist $n,\bar k$-independent $C_*,c_*>0$ and $\bar a\in\mathbb{R}^q$ such that

$$\kappa_{\bar k}\le\begin{cases} C_*e^{-c_*(\bar k-\bar k^*,\bar k-\bar k^*)+(\bar a,\bar k-\bar k^*)}, & |\bar k-\bar k^*|\le\widetilde\varepsilon n,\\ C_*e^{-c_0n^2} & |\bar k-\bar k^*|\ge\widetilde\varepsilon n.\end{cases} \qquad (3.18)$$

The proof of the first bound is based on Theorem 1. We write

$$\begin{aligned}
\log\Big(k^*_\alpha!Q_{k_\alpha,\beta}/k_\alpha!Q_{k^*_\alpha,\beta}\Big) &= \log\Big(Q_{k_\alpha;\beta}/Q^*_{k_\alpha;\beta}\Big)-\log\Big(Q_{k^*_\alpha,\beta}/Q^*_{k^*_\alpha,\beta}\Big) \qquad (3.19)\\
&\quad+\log\Big(k^*_\alpha!Q^*_{k_\alpha,\beta}/k_\alpha!Q^*_{k^*_\alpha,\beta}\Big),
\end{aligned}$$

where $Q^*_{k_\alpha;\beta}$ is defined in (1.32). Then (1.31) yields

$$\begin{aligned}
&\log\Big(Q_{k_\alpha,\beta}/Q^*_{k_\alpha,\beta}\Big)=\frac{k_\alpha^2\beta}{2}\Big(\mathcal{E}_{n/k_\alpha}+\frac34\Big)-k_\alpha I(k_\alpha)+O(1), \qquad (3.20)\\
&I(k_\alpha):=\frac{\beta}{4\pi i}\int_0^1dt\oint\Big(\frac{n}{k_\alpha}V_a(\zeta)-V^{(0)}_{n/k_\alpha}(\zeta)\Big)u_{n/k_\alpha}(\zeta,t)d\zeta+\Big(1-\beta/2\Big)\log(d_{n/k_\alpha}/2),
\end{aligned}$$

where $\mathcal{E}_{n/k_\alpha}=\mathcal{E}[(n/k_\alpha)V_\alpha^{(a)}]$ is the equilibrium energy (1.6), $u_{n/k_\alpha}(.,t)$, $V^{(0)}_{n/k_\alpha}$, and d_{n/k_α} are defined in (1.31). Using (1.30) and (2.31), we get for $I(k_\alpha)$ of (3.20)

$$k_\alpha I(k_\alpha)-k^*_\alpha I(k^*_\alpha)=b^{(\alpha)}(k_\alpha-k^*_\alpha)+O(1)+O((k_\alpha-k^*_\alpha)^2/n), \qquad (3.21)$$

where the concrete form of $b^{(\alpha)}$ is not important for us. Moreover, formula (1.32) for $n=$

$k_\alpha = k_\alpha^* + l$ and $n = k_\alpha^*$ yields

$$\begin{aligned}
&\log\Big(k_\alpha^*!Q^*_{k_\alpha,\beta}/k_\alpha!Q^*_{k^*_\alpha,\beta}\Big) - l\log\Big(\sqrt{2\pi}/\Gamma(\beta/2)\Big)\\
&\qquad = -\Big(\frac{\beta}{4}(k_\alpha^*+l)^2 + \frac{k_\alpha^*+l}{2}(1-\beta/2)\Big)\log\frac{\beta(k_\alpha^*+l)}{2}\\
&\qquad +\Big(\frac{\beta}{4}(k_\alpha^*)^2 + \frac{k_\alpha^*}{2}(1-\beta/2)\Big)\log\frac{\beta k_\alpha^*}{2} + \sum_{j=1}^{l}\log\Gamma(\beta(k_\alpha^*+j)/2)\\
&\qquad = l(1-\beta/2)\Big(\log(\beta k_\alpha^*/2) + \frac{1}{2}\Big) - \frac{3}{4}\beta k_\alpha^* l - \frac{3}{8}\beta l^2 + O(l^3/k_\alpha^*).
\end{aligned}\tag{3.22}$$

Here we used the Stirling formula

$$\begin{aligned}
&\log\Gamma(\beta(k_\alpha^*+j)/2) = \Big(\beta(k_\alpha^*+j)/2 - 1/2\Big)\log(\beta(k_\alpha^*+j)/2) - \beta(k_\alpha^*+j)/2 + O(n^{-1})\\
&= \Big(\beta(k_\alpha^*+j)/2 - 1/2\Big)\log(\beta k_\alpha^*/2) + \Big(\beta(k_\alpha^*+j)/2 - 1/2\Big)\frac{j}{k_\alpha^*} - \beta(k_\alpha^*+j)/2 + O(l/k_\alpha)
\end{aligned}$$

and the representation

$$\log\frac{\beta(k_\alpha^*+l)}{2} = \log\frac{\beta k_\alpha^*}{2} + \frac{l}{k_\alpha^*} - \frac{l^2}{2(k_\alpha^*)^2} + O(l^3/k_\alpha^3).$$

Moreover,

$$\frac{3k_\alpha^2\beta}{8} - \frac{(3k_\alpha^*)^2\beta}{8} = \frac{3\beta l k_\alpha}{4} + \frac{3\beta l^2}{8}.$$

The last relation and (3.19) – (3.22) yield

$$\begin{aligned}
\log\Big(k_\alpha^*!Q_{k_\alpha,\beta}/k_\alpha!Q_{k_\alpha^*,\beta}\Big) &= \frac{k_\alpha^2\beta}{2}\mathcal{E}_{n/k_\alpha} - \frac{(k_\alpha^*)^2\beta}{2}\mathcal{E}_{n/k_\alpha^*} + (k_\alpha - k_\alpha^*)a^{(\alpha)}\\
&+(k_\alpha-k_\alpha^*)(1-\beta/2)\log n + O(1),
\end{aligned}\tag{3.23}$$

where

$$a^{(\alpha)} = (1-\beta/2)\Big(\log(\beta k_\alpha^*/2n) + \frac{1}{2}\Big) + \log\Big(\sqrt{2\pi}/\Gamma(\beta/2)\Big) + b^{(\alpha)}$$

with $b^{(\alpha)}$ of (3.21). To estimate the difference of the energies we introduce the densities

$$\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}(\lambda) = \frac{k_\alpha}{n}\mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\rho_{n/k_\alpha}(\lambda), \quad \rho^{(\alpha)}(\lambda) = \rho(\lambda)\mathbf{1}_{\sigma_\alpha}(\lambda).\tag{3.24}$$

Then

$$\begin{aligned}
k_\alpha^2\mathcal{E}_{n/k_\alpha} &= n^2\Big(L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}, \widetilde{\rho}^{(\alpha)}_{n/k_\alpha}\Big] - \big(V_a, \widetilde{\rho}^{(\alpha)}_{n/k_\alpha})\Big)\\
&= n^2\Big(L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}, \widetilde{\rho}^{(\alpha)}_{n/k_\alpha}\Big] - \big(V, \widetilde{\rho}^{(\alpha)}_{n/k_\alpha}\big) + 2L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}, \rho - \rho^{(\alpha)}\Big]\Big).
\end{aligned}$$

Hence taking

$$n^2\mathcal{E}^{(\alpha)} := n^2\Big(L\big[\rho^{(\alpha)}, \rho^{(\alpha)}\big] - \Big(V, \rho^{(\alpha)}\Big) + 2L\big[\rho^{(\alpha)}, \rho - \rho^{(\alpha)}\big]\Big),$$

we obtain

$$\begin{aligned}
k_\alpha^2\mathcal{E}_{n/k_\alpha} - n^2\mathcal{E}^{(\alpha)} &= n^2\Big(L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha} - \rho^{(\alpha)}, \widetilde{\rho}^{(\alpha)}_{n/k_\alpha} - \rho^{(\alpha)}\Big] + 2L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha} - \rho^{(\alpha)}, \rho^{(\alpha)}\Big]\\
&\quad +2L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha} - \rho^{(\alpha)}, \rho - \rho^{(\alpha)}\Big] - \big(V, \widetilde{\rho}^{(\alpha)}_{n/k_\alpha} - \rho^{(\alpha)}\big)\\
&\le n^2 L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha} - \rho^{(\alpha)}, \widetilde{\rho}^{(\alpha)}_{n/k_\alpha} - \rho^{(\alpha)}\Big] =: n^2 L_{n/k_\alpha}.
\end{aligned}\tag{3.25}$$

Here we used that in view of (1.8) and our assumption $v^*=0$, we have

$$v(\lambda)=\int\log|\lambda-\mu|\rho(\mu)d\mu-V(\lambda)=0,\ \lambda\in\sigma_\alpha,\qquad v(\lambda)<0,\ \lambda\notin\sigma_\alpha,$$
$$\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}(\lambda)-\rho^{(\alpha)}(\lambda)=\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}(\lambda)\ge 0,\quad \lambda\notin\sigma_\alpha$$

which implies

$$L\Big[\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}-\rho^{(\alpha)},\rho\Big]-\big(V,\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}-\rho^{(\alpha)}\big)=\big(v,\widetilde{\rho}^{(\alpha)}_{n/k_\alpha}-\rho^{(\alpha)}\big)\le 0. \tag{3.26}$$

Note that for any function $\varphi:\operatorname{supp}\varphi\subset[-1,1],\int\varphi=\varphi^*$

$$L\Big[\varphi,\varphi\Big]=L\Big[\varphi-\varphi^*\psi_*,\varphi-\varphi^*\psi_*\Big]-|\varphi^*|^2\log 2\le-|\varphi^*|^2\log 2, \tag{3.27}$$

where $\psi_*(\lambda)=\pi^{-1}(1-\lambda^2)^{-1/2}\mathbf{1}_{[-1,1]}$, and we used the well known properties of ψ_*:

$$\int_{-1}^{1}\log|\lambda-\mu|\psi_*(\mu)d\mu=-\log 2,\ \lambda\in[-1,1],\qquad \int_{-1}^{1}\psi_*(\mu)d\mu=1.$$

Since, by (3.24),

$$\int\rho^{(\alpha)}_{n/k_\alpha}(\lambda)d\lambda=k_\alpha/n,\quad \int\rho^{(\alpha)}(\lambda)d\lambda=\mu^*_\alpha,$$

we get

$$n^2L_{n/k_\alpha}\le-\log 2|k_\alpha/n-\mu^*_\alpha|^2n^2. \tag{3.28}$$

In view of (3.23) and (3.25), to obtain the estimate for the difference of energies, it suffices to obtain the bound for $(k^*_\alpha)^2\mathcal{E}_{n/k^*_\alpha}-n^2\mathcal{E}^{(\alpha)}$ from below. But it follows from (2.31) and (1.23) that

$$\widetilde{\rho}^{(\alpha)}_{n/k^*_\alpha}(\lambda)-\rho^{(\alpha)}(\lambda)=O(|d^*_\alpha|^{1/2}),\quad v(\lambda)=O(|d^*_\alpha|^{3/2}),\quad \lambda\notin\sigma_\alpha, \tag{3.29}$$

where

$$d^*_\alpha:=k^*_\alpha/n-\mu^*_\alpha.$$

Since $|d^*_\alpha|\le 2n^{-1}$, the l.h.s. of (3.26) is $O(|d^*_\alpha|^3)=O(n^{-3})$ for $k_\alpha=k^*_\alpha$. Hence

$$(k^*_\alpha)^2\mathcal{E}_{n/k^*_\alpha}-n^2\mathcal{E}^{(\alpha)}=n^2L_{n/k^*_\alpha}+O(n^{-1}).$$

In order to estimate L_{n/k^*_α}, observe that if we denote $v_{n/k^*_\alpha}(\lambda)$ the l.h.s. of (1.8) for $(n/k^*_\alpha)V$, then we have

$$v_{n/k^*_\alpha}(\lambda)-v(\lambda)=d^*_\alpha V^{(a)}_\alpha(\lambda)+v^*_{n/k^*_\alpha},\quad \lambda\in\sigma_\alpha\cap\operatorname{supp}\widetilde{\rho}^{(\alpha)}_{n/k^*_\alpha}.$$

where

$$v^*_{n/k^*_\alpha}=\sup v_{n/k^*_\alpha}(\lambda)=v_{n/k^*_\alpha}(c_\alpha)=O(d^*_\alpha),$$

and c_α is a middle point of $\operatorname{supp}\widetilde{\rho}^{(\alpha)}_{n/k^*_\alpha}$. Moreover, the above argument implies that the contribution of the integrals over the domain $\sigma_{\alpha,\varepsilon}\setminus(\sigma_\alpha\cap\operatorname{supp}\widetilde{\rho}^{(\alpha)}_{n/k^*_\alpha})$ is $O(|d^*_\alpha|^{5/2})$. Thus

$$n^2L_{n/k^*_\alpha}=O(1),\quad\text{and}\quad k^2_\alpha\mathcal{E}_{n/k_\alpha}-(k^*_\alpha)^2\mathcal{E}_{n/k^*_\alpha}\le-\log 2(k_\alpha-k^*_\alpha)^2+O(1). \tag{3.30}$$

Then we obtain the first bound of (3.18) from (3.23) and the last inequality, if take the sum with respect to $\alpha=1,\dots,q$ and take into account that $\sum_\alpha(k_\alpha-k^*_\alpha)=0$.

To obtain the second bound of (3.18), we use the inequality

$$\log Q_{k_\alpha,\beta} \le \frac{\beta k_\alpha^2}{2}\mathcal{E}_{n/k_\alpha} + Ck_\alpha \log k_\alpha$$

with some universal C (see [2]). Using the inequality for k_α and k_α^*, we get in view of (3.30)

$$\log\Big(Q_{k_\alpha,\beta}/Q_{k_\alpha^*,\beta}\Big) \quad \le \quad -\frac{\beta}{2}\log 2|k_\alpha - k_\alpha^*|^2 + O(n\log n).$$

Hence we obtain the second inequality of (3.18). Now we are ready to find a bound for the l.h.s. of (3.9). We have

$$\begin{aligned}
\langle \Delta H\rangle_{H^a,\beta} \quad &= \sum_{k_1+\dots+k_q=n} \kappa_{\bar k}\, \langle \Delta H\rangle_{\bar k}\Big(\sum_{k_1+\dots+k_q=n}\kappa_{\bar k}\Big)^{-1}, \qquad (3.31)\\
\langle \Delta H\rangle_{\bar k} \quad &:= \sum_{\alpha\neq\alpha'} k_\alpha k_{\alpha'} L\Big[p_{1,\beta}^{(k_\alpha)}, p_{1,\beta}^{(k_{\alpha'})}\Big] - 2n\sum_\alpha k_\alpha\left(\widetilde V(\lambda), p_{1,\beta}^{(k_\alpha)}\right)\\
&\quad + n^2\sum_{\alpha\neq\alpha'} L\Big[\rho^{(\alpha)},\rho^{(\alpha')}\Big]\\
&= \sum_{\alpha\neq\alpha'} k_\alpha k_{\alpha'} L\Big[p_{1,\beta}^{(k_\alpha)} - \frac{n}{k_\alpha}\rho^{(\alpha)}, p_{1,\beta}^{(k_{\alpha'})} - \frac{n}{k_{\alpha'}}\rho^{(\alpha')}\Big].
\end{aligned}$$

Then, for $|\bar k - \bar k^*| \le \widetilde\varepsilon n$ we write similarly to (3.12) and (3.14)

$$\begin{aligned}
&\left|L\Big[p_{1,\beta}^{(k_\alpha)} - \frac{n}{k_\alpha}\rho^{(\alpha)}, p_{1,\beta}^{(k_{\alpha'})} - \frac{n}{k_{\alpha'}}\rho^{(\alpha')}\Big]\right|\\
&= \left|\sum_{j=-\infty}^{\infty} c_j\big(p_{1,\beta}^{(k_\alpha)} - \frac{n}{k_\alpha}\rho^{(\alpha)}, e^{ij\pi\lambda/3}\big)\big(p_{1,\beta}^{(k_{\alpha'})} - \frac{n}{k_{\alpha'}}\rho^{(\alpha')}, e^{ij\pi\lambda/3}\big)\right|\\
&\le \sum_{j=-\infty}^{\infty}|c_j|\bigg(\Big|\big(p_{1,\beta}^{(k_\alpha)} - \rho_{n/k_\alpha}, e^{ij\pi\lambda/3}\big)\Big|^2 + \Big|\big(p_{1,\beta}^{(k_{\alpha'})} - \rho_{n/k_{\alpha'}}, e^{ij\pi\lambda/3}\big)\Big|^2\\
&+\Big|\big(\rho_{n/k_\alpha} - \frac{n}{k_\alpha}\rho^{(\alpha)}, e^{ij\pi\lambda/3}\big)\Big|^2 + \Big|\big(\rho_{n/k_{\alpha'}} - \frac{n}{k_{\alpha'}}\rho^{(\alpha')}, e^{ij\pi\lambda/3}\big)\Big|^2\bigg)\\
&\le O(n^{-2}) + C\big(n/k_\alpha - (\mu_\alpha^*)^{-1}\big)^2 + \big(n/k_{\alpha'} - (\mu_{\alpha'}^*)^{-1}\big)^2.
\end{aligned}$$

To get the bound $O(n^{-2})$ for the first two terms in the r.h.s. here, we used (1.29) for $\varphi_k = e^{ik\pi\lambda/3}$ and the bound for c_j from (3.12), and for the last two terms we used (3.12), combined with the estimate

$$\Big|\big(\rho_{n/k_\alpha} - \frac{n}{k_\alpha}\rho^{(\alpha)}, e^{ij\pi\lambda/3}\big)\Big| \le C|n/k_\alpha - (\mu_\alpha^*)^{-1}|.$$

The estimate follows from (1.23) and (2.31), since these relations mean that ρ_η is differentiable with respect to η at $\eta = 1$, and $(\rho_\eta)'_\eta \in L_1[\sigma_{\alpha,\varepsilon}]$. For $|\bar k - \bar k^*| \ge \widetilde\varepsilon n$, in view of the second line of (3.18), it suffices to use that the r.h.s. of (3.31) is $O(n^2)$. Thus we get finally

$$|\,\langle \Delta H\rangle_{H^a,\beta}\,| \le C.$$

The bound, (1.20), and (3.9), yield

$$\det(T_n) = \left(\frac{Q_{n,1}Q_{n/2,4}}{Q_{n,2}(n/2)!2^n}\right)^2 \ge C\prod_{\alpha=1}^{q}\left(\frac{Q_{k_\alpha^*,1}Q_{k_\alpha^*/2,4}}{Q_{k_\alpha^*,2}(k_\alpha^*/2)!2^{k_\alpha^*}}\right)^2.$$

Hence it is enough to prove that each multiplier in the r.h.s. is bounded from below. Consider the functions V_t of (2.32). Then, as it was mentioned in the proof of Theorem 1 (iii), the limiting equilibrium density ρ_t has the form (2.34), which corresponds to V_t. Hence $V_{n/k^*_\alpha,t}$ satisfies conditions of Theorem 1 for any $t \in [0,1]$. Moreover, if we introduce the matrix $T_n(t)$ for the potential V_t by the same way, as above, then $T_n(0)$ corresponds to GOE or GSE. Consider the function

$$L(t) = \log\det T_{k^*_\alpha}(t). \tag{3.32}$$

To prove that $|L(1)| \le C$ it is enough to prove that

$$|L(0)| \le C, \quad |L'(t)| \le C, \quad t \in [0,1]. \tag{3.33}$$

The first inequality here follows from the results of [22]. To prove the second inequality we use (1.20) for V replaced by V_t. Then we get

$$\begin{aligned}
&L'(t) = (k^*_\alpha)^2\Big(\int \Delta V(\lambda) p^{(k^*_\alpha/2)}_{1,4,t}(\lambda)d\lambda + \int \Delta V(\lambda) p^{(k^*_\alpha)}_{1,1,t}(\lambda)d\lambda - 2\int \Delta V(\lambda) p^{(k^*_\alpha)}_{1,2,t}(\lambda)d\lambda\Big),\\
&\Delta V(\lambda) = (n/k^*_\alpha)V^{(a)}_\alpha(\lambda) - V^{(0)}_{n/k^*_\alpha}(\lambda).
\end{aligned}$$

Using (1.28), we obtain that the first and the second terms of (1.28) give zero contributions in $L'(t)$, hence $L'(t) = O(1)$. Thus we have proved the second inequality in (3.33) and so assertion (iii) of Lemma 2. As it was mentioned above, assertions (i) and (iii) of Lemma 2, combined with (3.3), imply that $F^{(1)}_{jk}$ and $F^{(4)}_{jk}$ of (1.18) are bounded uniformly in n.

To prove the first line of (1.37) for $\beta = 1$, we use (1.18)

$$\begin{aligned}
\int p^{(n)}_{1,1}(\lambda)f(\lambda)d\lambda &= n^{-1}\int S_{n,1}(\lambda,\lambda)f(\lambda)d\lambda\\
&= \int p^{(n)}_{1,2}(\lambda)f(\lambda)d\lambda + \sum_{j,k=-(2m-1)}^{2m-1} F^{(1)}_{jk}\int \psi^{(n)}_{n+j}(\lambda)\epsilon\psi^{(n)}_{n+k}(\lambda)f(\lambda)d\lambda.
\end{aligned}$$

Thus we obtain the first line of (1.37) from (3.15) and (3.5)(i). For $\beta = 4$ the proof is the same.

The second line of (1.37) follows from the first one in view of Lemma 1.

To obtain (1.38), observe that we proved already that the l.h.s. of (1.38) is more than the r.h.s. Hence we are left to prove the opposite inequality. To this aim we use the inequality (3.9) and the relation (3.10). Then each term in the first sum of (3.10) can be estimated by the same way, as for $\beta = 2$ by using the second line of (1.37) instead of (3.13). Each term in the second sum in (3.10) can be estimated by the same way, as for $\beta = 2$, if we use the first line of (1.37). □

Proof of Theorem 3. Convergence of the diagonal entries of $K_{n,1}$ and $K_{n/2,4}$ to the correspondent limiting expressions follows from the first assertion of Theorem 2, (ii) of (3.5) and convergence of $K_{n,2}$ to K_∞. The same is valid for 12-entries of $K_{n,1}$ and $K_{n/2,4}$. Moreover, since $\epsilon S_{n,\beta}(\lambda,\mu) = -\epsilon S_{n,\beta}(\mu,\lambda)$, one has

$$(\epsilon S_{n,1})(\lambda,\mu) = -\int_\lambda^\mu S_{n,1}(t,\mu)\,dt, \qquad (\epsilon S_{n/2,4})(\lambda,\mu) = -\int_\lambda^\mu S_{n/2,4}(t,\mu)\,dt,$$

which implies convergence of 21-entries of $K_{n,1}$ and $K_{n/2,4}$.□

4 Uniform bounds for integrals with $\epsilon(f\psi_k^{(n)})$

Set

$$\delta_n = n^{-2/3+\kappa},\ 0<\kappa<1/3, \tag{4.1}$$

$$\sigma_{\pm\delta_n} = \bigcup_{\alpha=1}^{q} \sigma_{\alpha,\pm\delta_n},\ \sigma_{\alpha,\pm\delta_n} = [E_{2\alpha-1}\mp\delta_n, E_{2\alpha}\pm\delta_n],\ (\sigma_{\alpha,-\delta_n}\subset\sigma_\alpha\subset\sigma_{\alpha,+\delta_n}).$$

Then, according to [5], we have

$$\begin{aligned}\psi_n^{(n)}(\lambda) &= R_0(\lambda)\cos n\pi F_n(\lambda)\ (1+O(n^{-1})), \quad \lambda\in\sigma_{-\delta_n},\\ \psi_{n-1}^{(n)}(\lambda) &= R_1(\lambda)\sin n\pi F_{n-1}(\lambda)\ (1+O(n^{-1})), \quad \lambda\in\sigma_{-\delta_n},\end{aligned} \tag{4.2}$$

where $R_0(\lambda)$, $R_1(\lambda)$ are some smooth functions, which may behave like $|X^{-1/4}(\lambda)|$ near each E_α (see (1.24) for the definition of X).

$$F_n(\lambda) = F(\lambda) + \frac{1}{n}m_0(\lambda), \quad F(\lambda) = \int_\lambda^{E_{2q}} \rho(\mu)d\mu, \quad F_{n-1}(\lambda) = F(\lambda) + \frac{1}{n}m_1(\lambda), \tag{4.3}$$

with ρ of (1.23) and smooth m_0, m_1, such that their first derivatives are bounded by $|X^{-1/2}(\lambda)|$. It will be important for us that

$$a_n^{(n)} R_0(\lambda)R_1(\lambda)\cos(\pi(m_0(\lambda)-m_1(\lambda))) = 1, \tag{4.4}$$

where $a_n^{(n)}$ is defined in (3.1). The relation follows from the fact (see [5]) that

$$\pi a_n^{(n)} n^{-1}\Big((\psi_n^{(n)}(\lambda))'\psi_{n-1}^{(n)}(\lambda) - \psi_n^{(n)}(\lambda)(\psi_{n-1}^{(n)}(\lambda))'\Big) = \rho(\lambda) + O(n^{-1})$$

For $|\lambda - E_\alpha|\le\delta_n$ we have

$$\begin{aligned}\psi_n^{(n)}(\lambda) &= n^{1/6}B_{11}^{(\alpha)}Ai\left(n^{2/3}\Phi_\alpha\Big((-1)^\alpha(\lambda-E_\alpha)\Big)\right)(1+O(|\lambda-E_\alpha|))\\ &\quad + n^{-1/6}B_{12}^{(\alpha)}Ai'\left(n^{2/3}\Phi_\alpha\Big((-1)^\alpha(\lambda-E_\alpha)\Big)\right)(1+O(|\lambda-E_\alpha|))+O(n^{-1}),\\ \psi_{n-1}^{(n)}(\lambda) &= n^{1/6}B_{21}^{(\alpha)}Ai\left(n^{2/3}\Phi_\alpha\Big((-1)^\alpha(\lambda-E_\alpha)\Big)\right)(1+O(|\lambda-E_\alpha|))\\ &\quad + n^{-1/6}B_{22}^{(\alpha)}Ai'\left(n^{2/3}\Phi_\alpha\Big((-1)^\alpha(\lambda-E_\alpha)\Big)\right)(1+O(|\lambda-E_\alpha|))+O(n^{-1}).\end{aligned} \tag{4.5}$$

Moreover,

$$|\psi_n^{(n)}(\lambda)| + |\psi_{n-1}^{(n)}(\lambda)| \le e^{-nc\,\mathrm{dist}^{3/2}\{\lambda,\sigma\}}, \quad \lambda\in\mathbb{R}\setminus\sigma_{-\delta_n}.$$

Functions Φ_α in (4.2) are analytic in some neighborhood of 0 and such that $\Phi_\alpha(\lambda) = a_\alpha x + O(x^2)$ with some positive a_α.

The proof of Lemma 2 is based on the proposition:

Proposition 1 *Under conditions of **C1-C3** for any smooth function f we have uniformly in $\sigma_{+\delta_n}$*

$$\begin{aligned}\epsilon(f\psi_n^{(n)})(\lambda) &= f(\lambda)R_0(\lambda)\frac{\cos nF_n(\lambda)}{nF_n'(\lambda)}\mathbf{1}_{\sigma_{-\delta_n}} + \chi_n(\lambda)\\ &\quad + \sum_{\alpha=1}^{2q}\left(f(E_\alpha)\frac{n^{-1/2}B_{11}^{(\alpha)}}{(-1)^\alpha\Phi_\alpha'(0)}\Psi\left(n^{2/3}\Phi_\alpha\Big((-1)^\alpha(\lambda-E_\alpha)\Big)\right)\right.\\ &\quad \left. + O(n^{-5/6})\right)\mathbf{1}_{|\lambda-E_\alpha|\le\delta_n} + \epsilon r_n(\lambda) + O(n^{-1}),\\ \Psi(x) &:= \int_{-\infty}^{x} Ai(t)dt,\end{aligned} \tag{4.6}$$

where $|\chi_n(\lambda)| \le n^{-1/2}C$ is a piecewise constant function which is a constant in each $\sigma_{\alpha,-\delta_n}$ and each interval $(E_\alpha - \delta_n, E_\alpha + \delta_n)$, and the remainder $r_n(\lambda)$ admits the bound

$$\int_{\sigma_{+\delta_n}} |r_n(\lambda)| d\lambda \le C n^{-1/2-3\kappa/4}. \tag{4.7}$$

Similar representation is valid for $\epsilon(f\psi_{n-1}^{(n)})$ if we replace sin *by* cos, *R_0 by R_1, F_n by F_{n-1}, and $B_{11}^{(\alpha)}$ by $B_{21}^{(\alpha)}$.*

Proof. Let $\lambda \in \sigma_{\alpha,-\delta_n}$. Then, integrating by parts in (4.2), we obtain

$$\int_{E_{2\alpha-1}+\delta_n}^{\lambda} f(\mu)\psi_n^{(n)}(\mu) d\mu = f(\mu)R(\mu)\frac{\sin nF_n(\mu)}{nF'(\mu)}\Big|_{E_{2\alpha-1}+\delta_n}^{\lambda}$$
$$- \int_{E_{2\alpha-1}+\delta_n}^{\lambda} \sin nF_n(\mu) \frac{d}{d\mu}\frac{f(\mu)R(\mu)}{nF'(\mu)} d\mu. \tag{4.8}$$

Moreover, by (4.5), we have for $|\lambda - E_{2\alpha}| \le \delta_n$

$$\int_{E_{2\alpha}-\delta_n}^{\lambda} \psi_n^{(n)}(\mu) f(\mu) d\mu = n^{-1/2} B_{11}^{(2\alpha)} f(\lambda) \frac{\Psi\left(n^{2/3}\Phi_{2\alpha}(\lambda - E_{2\alpha})\right) - \Psi\left(n^{2/3}\Phi_{2\alpha}(-\delta_n)\right)}{\Phi'(\lambda - E_\alpha)}$$
$$+O(n^{-5/6}) = n^{-1/2} B_{11}^{(2\alpha)} f(E_{2\alpha}) \frac{\Psi\left(n^{2/3}\Phi_{2\alpha}(\lambda - E_{2\alpha})\right)}{\Phi'(0)} + o(n^{-1/2})\mathrm{const} + O(n^{-5/6}).$$

Similar relations are valid for integrals near $E_{2\alpha-1}$. Taking into account that the integrals over $\mathbb{R} \setminus \sigma_{+\delta_n}$ are of the order $O(e^{-nc\delta_n^{3/2}}) = O(e^{-n^{3\kappa/2}})$, we obtain (4.6), writing $\epsilon(f\psi_n^{(n)})$ as a sum of the above integrals and similar ones (with integration from λ to $E_{2\alpha}$). Then, for $\lambda \in \sigma_{\alpha,-\delta_n}$

$$\chi_n(\lambda) = \frac{1}{2}\int_{E_1-\delta_n}^{E_{2\alpha-1}+\delta_n} f(\mu)\psi_n^{(n)}(\mu) d\mu - \frac{1}{2}\int_{E_{2\alpha-1}+\delta_n}^{E_{2q}+\delta_n} f(\mu)\psi_n^{(n)}(\mu) d\mu = O(n^{-1/2}),$$

and r_n is the sum of the terms, which are under the integrals in the r.h.s. of (4.8).

Hence we are left to prove the bound for r_n. Using that

$$F_n'(\lambda) = (2\pi)^{-1} P(\lambda) X^{1/2}(\lambda) + n^{-1} m'(\lambda), \quad |R(\lambda)| \le C|X^{-1/4}(\lambda)|,$$

we have

$$\begin{aligned} \int |r_n(\mu)| d\mu &\le \int_{\sigma_{-\delta_n}} \left| \frac{d}{d\mu} \frac{f(\mu)R(\mu)}{nF'(\mu)} \right| d\mu + O(n^{-5/6}) \\ &\le C(1 + \|f'\|_\infty) n^{-1} \delta_n^{-3/4} = C(1 + \|f'\|_\infty) n^{-1/2-3\kappa/4}. \end{aligned}$$

□

Proof of Lemma 2. Using recursion relations (3.1), it is easy to get that for any $|j| \le 2m$

$$\psi_{n+j}^{(n)}(\lambda) = f_{0j}(\lambda)\psi_n^{(n)}(\lambda) + f_{1j}(\lambda)\psi_{n-1}^{(n)}(\lambda),$$

where f_{0j} and f_{1j} are polynomials of degree at most $|j|$. Note that since $a_k^{(n)}$ and $b_k^{(n)}$ are bounded uniformly in n for $k - n = o(n)$, f_{0j} and f_{1j} have coefficients, bounded uniformly

in n. Hence assertion (ii) follows from Proposition 1. Moreover, it follows from the above argument that to prove assertion (i) it suffices to estimate

$$I_1 := (g\psi_{n-1}^{(n)}, \epsilon(f\psi_n^{(n)})),\ I_2 := (g\psi_{n-1}^{(n)}, \epsilon(f\psi_{n-1}^{(n)})),\ I_3 := (g\psi_n^{(n)}, \epsilon(f\psi_n^{(n)})). \tag{4.9}$$

with differentiable f, g. It follows from Proposition 1 that

$$I_1 = I_{1,0} + \sum_{\alpha=1}^{2q} I_{1,\alpha} + (g\psi_{n-1}^{(n)},\, \epsilon r_n) + O(n^{-1}),$$

where

$$\begin{aligned} I_{1,0} &= n^{-1}k_n \int_{\sigma_{-\delta_n}} f(\lambda)g(\lambda)R_0(\lambda)R_1(\lambda)\frac{\sin nF_n(\lambda)\sin nF_{n-1}(\lambda)}{F_n'(\lambda)}d\lambda \\ I_{1,2\alpha} &= n^{-1/3}B_{11}^{(2\alpha)}B_{21}^{(2\alpha)}f(E_{2\alpha})g(E_{2\alpha})\int_{E_{2\alpha}-\delta_n}^{E_{2\alpha}+\delta_n} \Psi\left(n^{2/3}\Phi_{2\alpha}(\lambda - E_{2\alpha})\right) \\ &\quad \cdot \frac{Ai\left(n^{2/3}\Phi_{2\alpha}(\lambda - E_{2\alpha})\right)}{\Phi_{2\alpha}'(0)}d\lambda, \end{aligned}$$

and $I_{1,2\alpha-1}$ is the integral similar to $I_{1,2\alpha}$ for the region $|\lambda - E_{2\alpha-1}| \le \delta_n$. It is easy to see that

$$I_{1,\alpha} = B_{11}^{(\alpha)}B_{21}^{(\alpha)}\frac{f(E_\alpha)g(E_\alpha)}{2n(\Phi_\alpha'(0))^2}\left(1 + o(1)\right),$$

Moreover, (4.7) and (ii) of Lemma 2 yield

$$(g\psi_{n-1}^{(n)}, \epsilon r_n) = -(\epsilon(g\psi_{n-1}^{(n)}), r_n) \le \int |\epsilon(g\psi_{n-1}^{(n)})|\,|r_n|\,d\lambda = O(n^{-1-3\kappa/4}).$$

Hence we are left to find the bound for $I_{1,0}$.

$$\begin{aligned} I_{1,0} &= (2n)^{-1}\int_{\sigma_{-\delta_n}} f(\lambda)g(\lambda)R_0(\lambda)R_1(\lambda)\frac{\cos\pi(m_0(\lambda) - m_1(\lambda))}{F_n'(\lambda)}d\lambda \\ &\quad + (2n)^{-1}\int_{\sigma_{-\delta_n}} f(\lambda)g(\lambda)R_0(\lambda)R_1(\lambda)\frac{\cos n(F_n(\lambda) + F_{n-1}(\lambda))}{F_n'(\lambda)}d\lambda = I_{10}' + I_{10}''. \end{aligned}$$

By (4.4) and (4.3), we obtain

$$I_{11}' = \frac{1 + o(1)}{2n}\int_{-2}^{2}\frac{f(\lambda)g(\lambda)}{P(\lambda)X^{1/2}(\lambda)}d\lambda.$$

Integrating by parts, one can obtain easily that $I_{11}'' = O(n^{-2}\delta_n^{-3/2}) = O(n^{-1-3\kappa/2})$.

The other two integrals from (4.9) can be estimated similarly.

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